Midterm #1 – Practice

MATH 126 — Calculus II
Midterm: Thursday, Sept 26, 2024

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. A nongraphing calculator (equivalent to the TI-30XIIS) is allowed on the exam (but not needed). No notes or further tools of any kind will be permitted on the midterm exam.

Problem 1. Go over all the quizzes!

To help you with that, there is a version of each quiz posted on our course website without solutions (of course, there are solutions, too).

Problem 2. Find the length of the following curve:

$$y = 1 - 2x^{3/2}, \quad 0 \leqslant x \leqslant \frac{1}{3}.$$

Solution. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = -3\sqrt{x}$, the length of the curve is given by

$$\int_0^{1/3} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x = \int_0^{1/3} \sqrt{1+9x}\, \mathrm{d}x = \left[\frac{2}{3} \cdot \frac{1}{9}(1+9x)^{3/2}\right]_0^{1/3} = \frac{2}{27}(8-1) = \frac{14}{27}.$$

Problem 3.

(a) Evaluate the integral $\displaystyle\int_0^2 \frac{x}{\sqrt{4-x^2}}\, \mathrm{d}x$.

(b) Evaluate the integral $\displaystyle\int_0^2 \frac{x^3}{\sqrt{4-x^2}}\, \mathrm{d}x$.

Solution.

(a) We substitute $u = 4 - x^2$ so that $\mathrm{d}u = -2x\,\mathrm{d}x$ to get (if $x = 0$ then $u = 4$; if $x = 2$ then $u = 0$)

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\, \mathrm{d}x = -\frac{1}{2}\int_4^0 \frac{\mathrm{d}u}{\sqrt{u}} = \left[-\frac{1}{2} \cdot 2\sqrt{u}\right]_4^0 = \sqrt{4} = 2.$$

(b) We again substitute $u = 4 - x^2$ so that $\mathrm{d}u = -2x\,\mathrm{d}x$ to get (if $x = 0$ then $u = 4$; if $x = 2$ then $u = 0$). Compared to the previous integral, there is x^2 remaining in the integrand which we substitute as $x^2 = 4 - u$. Overall,

$$\begin{aligned}\int_0^2 \frac{x^3}{\sqrt{4-x^2}}\, \mathrm{d}x &= -\frac{1}{2}\int_4^0 \frac{x^2 \mathrm{d}u}{\sqrt{u}} = -\frac{1}{2}\int_4^0 \frac{4-u}{\sqrt{u}} \mathrm{d}u = -\frac{1}{2}\int_4^0 (4u^{-1/2} - u^{1/2}) \mathrm{d}u \\ &= -\frac{1}{2}\left[8u^{1/2} - \frac{2}{3}u^{3/2}\right]_4^0 = -\frac{1}{2}\left(0 - \left(8 \cdot 2 - \frac{2}{3} \cdot 8\right)\right) = \frac{16}{3}.\end{aligned}$$

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Problem 4. Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^2+1}, \quad y(0) = 2.$$

Solution. Using separation of variables we find

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int \frac{1}{x^2+1}\,\mathrm{d}x$$

and therefore

$$-\frac{1}{y} = \arctan(x) + C.$$

Using the initial value $y = 2$ when $x = 0$, we get $-\frac{1}{2} = \arctan(0) + C = C$.

Finally, using $C = -\frac{1}{2}$ and solving for y, we find that the solution to the initial value problem is

$$y(x) = -\frac{1}{\arctan(x) - \frac{1}{2}}.$$

Problem 5. Consider the region (in the first quadrant) enclosed by the curves

$$y = \frac{1}{x}, \quad y = \frac{1}{x^2}, \quad x = 2.$$

(a) Sketch the region and find its area.

(b) Find the volume of the solid obtained by revolving this region about the line $y = 0$.

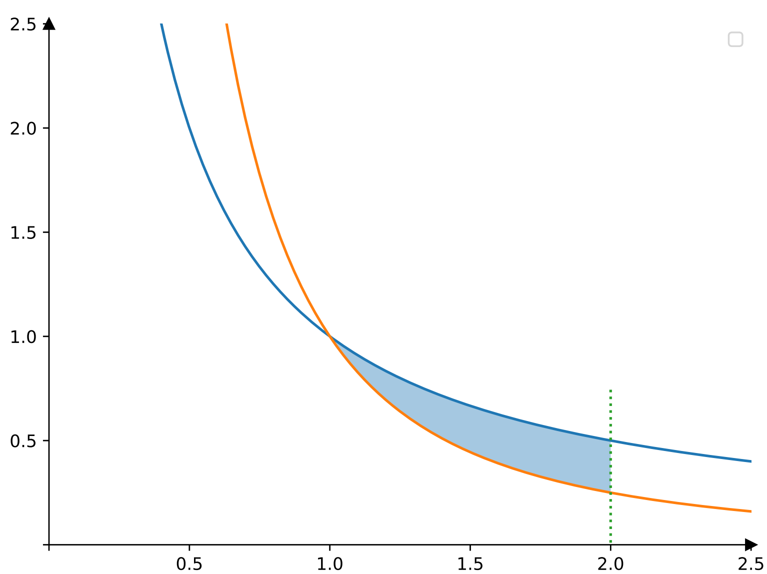
(c) Find the volume of the solid obtained by revolving this region about the line $y = -1$.

(d) Using the shell method, find the volume of the solid obtained by revolving this region about the y-axis.

(e) Using the washer method, compute the same volume.

Solution.

(a) Here is a sketch:



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The two curves intersect at $x=1$ (this is visible from the sketch, or follows from equating $\frac{1}{x}=\frac{1}{x^2}$). The region therefore extends from $x=1$ to $x=2$. In that interval, $\frac{1}{x}$ is bigger than $\frac{1}{x^2}$. Therefore, the area is given by

$$\int_1^2 \left(\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x = \left[\ln|x|+\frac{1}{x}\right]_1^2 = \ln(2)+\frac{1}{2}-1 = \ln(2)-\frac{1}{2}.$$

(b) We are revolving about the x-axis. Using the washer method, the volume is

$$\int_1^2 \left[\pi\left(\frac{1}{x}\right)^2-\pi\left(\frac{1}{x^2}\right)^2\right]\mathrm{d}x = \pi\int_1^2\left(\frac{1}{x^2}-\frac{1}{x^4}\right)\mathrm{d}x = \pi\left[-\frac{1}{x}+\frac{1}{3}\frac{1}{x^3}\right]_1^2 = \pi\left(-\frac{11}{24}-\left(-\frac{2}{3}\right)\right) = \frac{5\pi}{24}.$$

(c) We are now revolving about the horizontal line $y=-1$. Using the washer method, the volume is

$$\begin{aligned}\int_1^2\left[\pi\left(\frac{1}{x}-(-1)\right)^2-\pi\left(\frac{1}{x^2}-(-1)\right)^2\right]\mathrm{d}x &= \pi\int_1^2\left[\left(\frac{1}{x}+1\right)^2-\left(\frac{1}{x^2}+1\right)^2\right]\mathrm{d}x\\ &= \pi\int_1^2\left[\frac{1}{x^2}+\frac{2}{x}+1-\left(\frac{1}{x^4}+\frac{2}{x^2}+1\right)\right]\mathrm{d}x\\ &= \pi\int_1^2\left[\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^4}\right]\mathrm{d}x\\ &= \pi\left[2\ln|x|+\frac{1}{x}+\frac{1}{3}\frac{1}{x^3}\right]_1^2 = \pi\left(2\ln(2)-\frac{19}{24}\right).\end{aligned}$$

(d) The shell at position x has radius x and height $\frac{1}{x}-\frac{1}{x^2}$. Given a thickness of $\mathrm{d}x$, it therefore has (approximate) volume $2\pi x(\frac{1}{x}-\frac{1}{x^2})\mathrm{d}x$. Hence, the total volume is

$$\int_1^2 2\pi x\left(\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x = 2\pi\int_1^2\left(1-\frac{1}{x}\right)\mathrm{d}x = 2\pi\Big[x-\ln|x|\Big]_1^2 = 2\pi(1-\ln(2)).$$

(e) Note that we need to consider horizontal slices from $y=\frac{1}{2^2}$ (look for the lowest point in our sketch) to $y=1$. There are two kinds of slices: first, from $y=\frac{1}{2^2}$ to $y=\frac{1}{2}$, these slices extend from the curve $y=\frac{1}{x^2}$ (so that $x=\frac{1}{\sqrt{y}}$) to the vertical line $x=2$. Second, from $y=\frac{1}{2}$ to $y=1$, these slices extend from the curve $y=\frac{1}{x^2}$ (so that $x=\frac{1}{\sqrt{y}}$) to the curve $y=\frac{1}{x}$ (so that $x=\frac{1}{y}$). With this in mind, the total volume is

$$\begin{aligned}&\int_{1/4}^{1/2}\left[\pi\cdot 2^2-\pi\left(\frac{1}{\sqrt{y}}\right)^2\right]\mathrm{d}y+\int_{1/2}^1\left[\pi\left(\frac{1}{y}\right)^2-\pi\left(\frac{1}{\sqrt{y}}\right)^2\right]\mathrm{d}y\\ =\;& \pi\int_{1/4}^{1/2}\left[4-\frac{1}{y}\right]\mathrm{d}y+\pi\int_{1/2}^1\left[\frac{1}{y^2}-\frac{1}{y}\right]\mathrm{d}y\\ =\;& \pi\Big[4y-\ln|y|\Big]_{1/4}^{1/2}+\pi\left[-\frac{1}{y}-\ln|y|\right]_{1/2}^1\\ =\;& \pi\left(2-\ln\left(\frac{1}{2}\right)-\left(1-\ln\left(\frac{1}{4}\right)\right)\right)+\pi\left(-1-\left(-2-\ln\left(\frac{1}{2}\right)\right)\right)\\ =\;& \pi\left(2+\ln\left(\frac{1}{4}\right)\right) = \pi(2-2\ln(2)),\end{aligned}$$

which matches what we computed (more easily!) using the shell method.

Problem 6. Evaluate the following indefinite integral: $\int x\cos(x^2+3)\,\mathrm{d}x$.

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Solution. We substitute $u = x^2 + 3$ (so that $\mathrm{d}u = 2x\,\mathrm{d}x$) to get

$$\int x\cos(x^2+3)\,\mathrm{d}x = \frac{1}{2}\int \cos(u)\mathrm{d}u = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2+3) + C.$$

Problem 7. Consider the region bounded by the curves

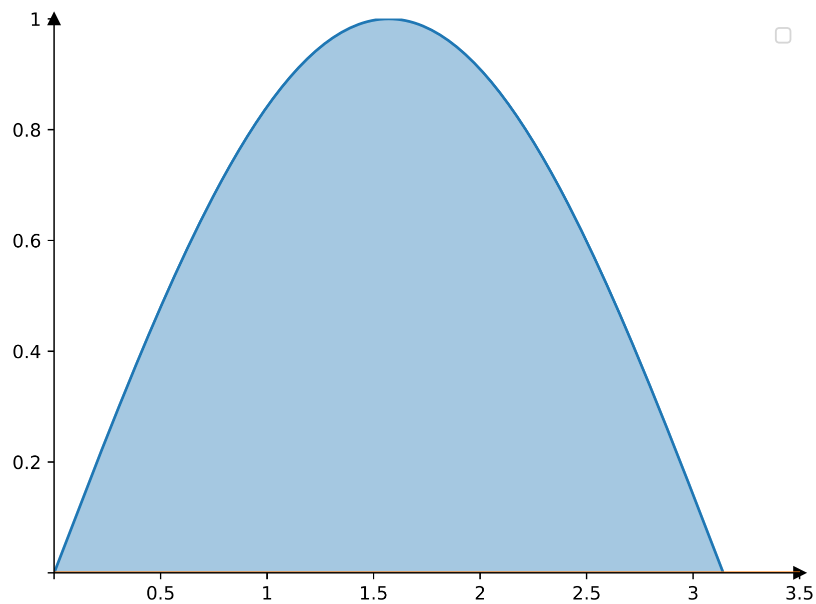
$$y = 0, \quad y = \sin(x), \quad 0 \leqslant x \leqslant \pi.$$

Sketch the region, then set up an integral (but do not evaluate it) for the volume of the solid obtained by rotating this region about the x-axis.

(a) Set up an integral (but do not evaluate it) for the volume of the solid obtained by rotating this region about the x-axis.

(b) Set up an integral (but do not evaluate it) for the volume of the solid obtained by revolving the region about the y-axis.

Solution. We should always start with a sketch:



(a) Using the disk/washer method, the volume is

$$\int_0^\pi \pi\sin^2(x)\,\mathrm{d}x.$$

(We can evaluate this integral using a trig identity or via integration by parts, a method that we soon learn about.)

(b) Using the shell method, the volume is

$$\int_0^\pi 2\pi x\sin(x)\mathrm{d}x.$$

(This, again, is an integral that can be evaluated via integration by parts.)

Problem 8. Consider the cylindrical container displayed to the right. It is half filled with sand weighing 100 lb/ft^3.

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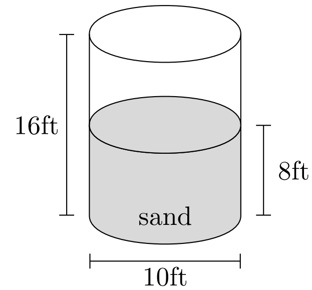
(a) Determine the amount of work needed to lift the sand to the rim of the tank.

(b) Determine the amount of work needed to lift the sand to a level 10 ft above the rim of the tank. Just an integral is good enough, here.

(c) Now, suppose the container is completely filled with sand. Determine the amount of work needed to lift the sand to a level 10 ft above the rim of the tank. Again, an integral is good enough, here.



Solution.

(a) Let x measure height (in ft) starting from the bottom of the container.

We consider a (horizontal) "slice" of the container at position x (and thickness dx).

- The volume of this slice is $\text{vol} = \pi \cdot 5^2 \cdot \mathrm{d}x$ (ft^3). Its weight is $2500\pi\,\mathrm{d}x$ (lb).
- This slice needs to be lifted $16 - x$ (ft).
- Thus, the work for this slice is $2500\pi(16 - x)\,\mathrm{d}x$ (ft-lb).

There are slices from $x = 0$ to $x = 8$. "Adding" these up, we find that the total amount of work is

$$\text{work} = \int_0^8 2500\pi(16 - x)\,\mathrm{d}x = 2500\pi\left[16x - \frac{1}{2}x^2\right]_0^8 = 2500\pi \cdot 96 \approx 754,000\,\text{ft-lb}.$$

(b) The only adjustment to the first part is that each slice needs to be lifted $16 + 10 - x$ (ft) now. Hence, we find that the total amount of work is

$$\text{work} = \int_0^8 2500\pi(26 - x)\,\mathrm{d}x = 2500\pi \cdot 176 \approx 1,382,000\,\text{ft-lb}.$$

(c) The only further adjustment is that there are now slices from $x = 0$ to $x = 16$. The total amount of work is

$$\text{work} = \int_0^{16} 2500\pi(26 - x)\,\mathrm{d}x = 2500\pi \cdot 288 \approx 2,262,000\,\text{ft-lb}.$$

Problem 9. A conical container of radius 15 ft and height 20 ft is completely filled with water (the tip of the cone is at the bottom). Write down an integral for how much work it will take to pump the water to a level of 10 ft above the cone's rim. Do not evaluate the integral. [Water weighs 62.4 lb/ft^3.]

Solution. Make a sketch! Let us denote with y (in ft) the vertical position in such a way that the tip is at $y = 0$ and the rim is at $y = 20$. We need to pump the water to the level $y = 20 + 10 = 30$.

We consider a horizontal slice at height y and thickness dy:

- This slice needs to be lifted up $30 - y$ (ft).
- The slice is (almost) a disk with radius $r = \frac{15}{20}y = \frac{3}{4}y$. Hence, its volume is $\pi r^2\,\mathrm{d}y = \frac{9\pi}{16}y^2\mathrm{d}y$ (ft^3).
- The weight of the slice is $62.4\,\frac{9\pi}{16}y^2\,\mathrm{d}y$ (lb) and it needs to be lifted up $30 - y$ (ft). That takes work of $(30 - y) \cdot 62.4\,\frac{9\pi}{16}y^2\mathrm{d}y$ (ft-lb).
- "Adding" up, the total work required is

$$\int_0^{20} (30 - y) \cdot 62.4\,\frac{9\pi}{16}y^2\mathrm{d}y,$$

which is about $4.41 \cdot 10^6$ ft-lb (but we were asked to not evaluate the integral).

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