**Open problem.** It is unknown whether the Flint Hills series $\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)}$ converges.

We expect that it converges: https://mathoverflow.net/questions/24579/

Note that $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^3}$ converges by comparison with the $p$-series with $p = 3 > 1$ (because $0 \leqslant \sin^2(x) \leqslant 1$).

Such a comparison does not work in the Flint Hills series because $\sin(n)$ can be close to $0$ (if $n$ is close to $\pi$, $2\pi, 3\pi, \ldots$).

**Example 183.** What is the difference between the parametric curves

$$x = \cos(t),\ y = \sin(t) \text{ with } t \in [0, 2\pi] \quad \text{and} \quad x = \cos(2\pi t),\ y = \sin(2\pi t) \text{ with } t \in [0, 1]?$$

**Solution.** It is the same curve (unit circle around the origin). Only the parametrization differs: in the second case, we "go through" the circle at $2\pi$ times the speed compared to the first case.

**Example 184.** Find the slope of the line tangent to the curve $x = \cos(t)$, $y = \sin(t)$ at the point corresponding to $t = \frac{\pi}{4}$. Also determine $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at that point.

**Solution. (slope via sketch)** Make a sketch! Since the curve is the unit circle and $\frac{\pi}{4} = 45°$, the slope is clearly $-1$. We cannot likewise see the exact value of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$. However, we can observe that the curve is concave down at our point so that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$.

**Solution.** The tangent line has slope $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\cos t}{-\sin t}$, which for $t = \frac{\pi}{4}$ is $\frac{\cos\frac{\pi}{4}}{-\sin\frac{\pi}{4}} = \frac{\sqrt{\frac{1}{2}}}{-\sqrt{\frac{1}{2}}} = -1$.

For $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, we write it as $\frac{\mathrm{d}y'}{\mathrm{d}x}$ where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ and compute it as

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cos t}{\sin t}}{-\sin t} = \frac{\frac{-\sin^2 t - \cos^2 t}{\sin^2 t}}{\sin t} = -\frac{1}{\sin^3 t}.$$

For $t = \frac{\pi}{4}$, this is $-\frac{1}{\sin^3(\frac{\pi}{4})} = -(\sqrt{2})^3 = -\sqrt{8}$.

**For comparison.** Using Cartesian coordinates, we have $y = \sqrt{1 - x^2}$ and the point in question is $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}}$$

When $x = \sqrt{\frac{1}{2}}$, the slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is $-\frac{\sqrt{\frac{1}{2}}}{\sqrt{1 - \frac{1}{2}}} = -1$.

For the second derivative,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{x}{\sqrt{1 - x^2}}\right) = -\frac{\sqrt{1 - x^2} - x\left(-\frac{x}{\sqrt{1 - x^2}}\right)}{1 - x^2}.$$

Without simplifying, we plug in $x = \sqrt{\frac{1}{2}}$ to find that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is $-\frac{\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}}(-1)}{\frac{1}{2}} = -4\sqrt{\frac{1}{2}} = -\sqrt{8}$.

**Note.** We can also work from other descriptions of the circle such as $x^2 + y^2 = 1$. Do it!

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**Example 185.** Consider the curve described, for $t$ in $[0, 4\pi]$, by the parametric equations

$$x = t\cos(t), \quad y = t\sin(t).$$

(a) Make a rough sketch of the curve.

Highlight the point corresponding to $t = \frac{\pi}{2}$ and sketch the tangent line through that point.

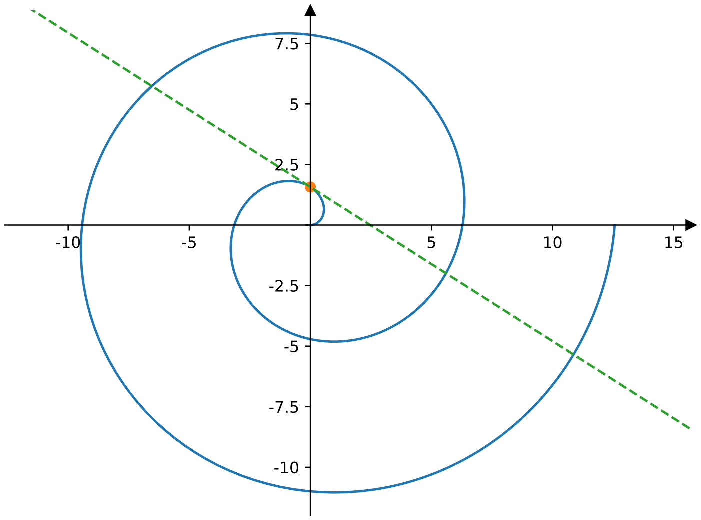
(b) Find the slope of the line tangent to the curve at the point corresponding to $t = \frac{\pi}{2}$.

(c) Setup an integral for the arc length of the curve.

(d) Determine $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the point corresponding to $t = \frac{\pi}{2}$.

**Solution.**

(a) This is a spiral starting at the origin (this is the point we get for $t = 0$) and completing exactly $2$ turns to end at the point $(4\pi, 0)$ on the $x$-axis (for $t = 4\pi$).




(b) The tangent line has slope $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$.

At our point, that is when $t = \frac{\pi}{2}$, the slope is $\frac{\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$.

(c) The arc length of our curve is

$$\int_0^{4\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{4\pi} \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}\,\mathrm{d}t = \int_0^{4\pi} \sqrt{1 + t^2}\,\mathrm{d}t$$

The final integral could be evaluated using $\int \sqrt{1 + t^2}\,\mathrm{d}t = \frac{1}{2}t\sqrt{1 + t^2} + \frac{1}{2}\log\left(t + \sqrt{1 + t^2}\right) + C$.

If you want a challenge, try to derive this indefinite integral using our techniques.

(d) We already computed that $y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$. We next compute $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$.

Finally, we plug in $t = \frac{\pi}{2}$. This calculation is a bit messy. The final answer should be $-\frac{2}{\pi} - \frac{16}{\pi^3}$.

**Comment.** From the plot, we knew that, at that point, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$ since the curve is visibly concave down.

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