Notes for Lecture 37 Mon, 11/25/2024

Review. Polar coordinates

Parametric curves

Example 180. The unit circle is described by the Cartesian equation $x^2 + y^2 = 1$ (in polar coordinates, the equation would be $r = 1$). Instead of such coordinate equations, we can also describe the same curve by **parametrizing** it: $x = \cos(t)$, $y = \sin(t)$ with parameter $t \in [0, 2\pi]$.

Comment. A curve can be parametrized in many ways. For instance, $x = t$, $y = \sqrt{1 - t^2}$ with $t \in [-1, 1]$ is another parametrization of the upper half-circle.

Remark. Note the difference in philosophies behind describing curves: an equation like $x^2 + y^2 = 1$ is "exclusionary" because we start with all points (x, y) and then restrict to those with $x^2 + y^2 = 1$. On the other hand, $x = \cos(t)$, $y = \sin(t)$ with $t \in [0, 2\pi]$ is "inclusionary" because we are listing precisely the points on the curve.

We can work with parametric curves similarly to what we have been doing. For instance:

(arc length) The parametric curve $x = f(t)$, $y = g(t)$ with $t \in [a, b]$ has arc length

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,\mathrm{d}t.$$

Note that $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$ equals $\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$ from earlier.

Example 181. Using the parametric curve $x = r\cos(t)$, $y = r\sin(t)$ with parameter $t \in [0, 2\pi]$, find the circumference of a circle of radius r.

Solution. $L = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{2\pi} \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}\,\mathrm{d}t = \int_0^{2\pi} r\,\mathrm{d}t = 2\pi r$

Given a parametric curve $x = f(t)$, $y = g(t)$, we can compute ordinary derivatives as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} \qquad \left[= \frac{g'(t)}{f'(t)}\right]$$

Likewise, writing $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y'}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} \qquad \left[= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{g'(t)}{f'(t)}\right)}{f'(t)} = \frac{g''(t)f'(t) - g'(t)f''(t)}{(f'(t))^3}\right]$$

Why? This is just the chain rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$ (in our case, $\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{f'(t)}$)

It tells us that we can replace $\frac{\mathrm{d}}{\mathrm{d}x}$ (the derivative with respect to x) with $\frac{\mathrm{d}}{\mathrm{d}t}$ if we multiply the result with $\frac{\mathrm{d}t}{\mathrm{d}x}$.

Example 182. Consider the parametric curve given by $x = t^2$, $y = t + 1$ with $t \geqslant 0$.

(a) Give an equivalent (non-parametric) Cartesian equation.

(b) Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at the point corresponding to $t = 1$.

Solution.

(a) If $x = t^2$, then $t = \sqrt{x}$, and so the curve is given by the Cartesian equation $y = \sqrt{x} + 1$.
Comment. In general, eliminating the parameter, as we did here, may be difficult or impossible.

(b) In order to see that we are really computing the same thing, we proceed both from the Cartesian equation $y = \sqrt{x} + 1$ as well as from the parametric equations:

- **(Cartesian equation)** Starting with $y(x) = \sqrt{x} + 1$, we have:

$$\begin{aligned} y'(x) &= \frac{1}{2\sqrt{x}} \quad \Longrightarrow \quad y'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} \\ y''(x) &= -\frac{1}{4} x^{-3/2} \quad \Longrightarrow \quad y''(1) = -\frac{1}{4} \end{aligned}$$

- **(parametric equations)** We now use $x = t^2$, $y = t + 1$ to compute the same quantities:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{1}{2t} \quad \Longrightarrow \quad \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{t=1} = \frac{1}{2} \\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-1/(2t^2)}{2t} = -\frac{1}{4t^3} \quad \Longrightarrow \quad \left[\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right]_{t=1} = -\frac{1}{4} \end{aligned}$$