Polar coordinates

Our usual coordinates (x, y) used to describe points in the plane are Cartesian coordinates. Polar coordinates are an alternative way of describing points.

The **polar coordinates** (r, θ) represent the point $(x, y) = r(\cos\theta, \sin\theta)$.

This means $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
Important comment. Often, θ is taken from $[0, 2\pi)$ (but $(-\pi, \pi]$ is another popular choice), and, usually, $r \geqslant 0$.

Example 174. Which point (in Cartesian coordinates) has polar coordinates $r = 2$, $\theta = \frac{\pi}{6}$?

Solution. $(x, y) = r(\cos\theta, \sin\theta) = 2\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = (\sqrt{3}, 1)$
[Draw a right triangle with angle $\frac{\pi}{6} = 30°$ to find $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$.]

Note. The polar coordinates $r = 2$, $\theta = \frac{\pi}{6} + 2\pi$ correspond to the same point $(\sqrt{3}, 1)$. Polar coordinates are not quite unique.
Note. Sometimes, we permit negative r. For instance, the polar coordinates $r = -2$, $\theta = \frac{\pi}{6} + \pi$ also describe the point $(\sqrt{3}, 1)$.

How to calculate the polar coordinates (r, θ) for (x, y)?
By Pythagoras, $r = \sqrt{x^2 + y^2}$, and the angle is $\theta = \operatorname{atan2}(y, x) \in (-\pi, \pi]$.

Why? It follows from $x = r\cos(\theta)$ and $y = r\sin(\theta)$ that $\frac{y}{x} = \tan(\theta)$. We therefore get $\theta = \arctan\left(\frac{y}{x}\right)$ if θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (plot $\tan$ and $\arctan$ to remind yourself that $\arctan$ only takes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$).
The function $\operatorname{atan2}$ is available in most programming languages (C, C++, PHP, Java, ...) and is a version of $\arctan(x)$ (or atan in those languages). Note that $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan(\theta)$. If our point is in the first or fourth quadrant, then $\theta = \arctan\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Otherwise, $\theta = \arctan\left(\frac{y}{x}\right) + \pi$ (see next example).

Example 175. Find the polar coordinates, with $r \geqslant 0$ and $\theta \in [0, 2\pi)$ of $(5, 5)$ and $(-5, -5)$.

Solution. First, plot both points!
The polar coordinates of $(5, 5)$ are $r = 2\sqrt{5}$ and $\theta = \frac{\pi}{4}$.
The polar coordinates of $(-5, -5)$ are $r = 2\sqrt{5}$ and $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Note. $(5, 5)$ is in the first quadrant and $\theta = \arctan\left(\frac{y}{x}\right) = \arctan(1) = \frac{\pi}{4}$. On the other hand, $(-5, -5)$ is in the third quadrant, and so $\theta = \arctan\left(\frac{y}{x}\right) + \pi = \arctan(1) + \pi = \frac{5\pi}{4}$. [$\operatorname{atan2}$ allows us to avoid this distinction.]

Example 176. Describe a circle around the origin with radius 3 using Cartesian and polar coordinates.

Solution. Using Cartesian coordinates, the circle is described by $x^2 + y^2 = 3^2$.
Using polar coordinates, the circle is described by the even simpler equation $r = 3$.

Note. In this case, both coordinate equations are easy to see directly. We can, however, convert any equation in Cartesian coordinates to polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$. In our case, we would go from $x^2 + y^2 = 3^2$ to $(r\cos\theta)^2 + (r\sin\theta)^2 = 3^2$, which simplifies to $r^2 = 9$ or $r = 3$ (if we work with $r \geqslant 0$).

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Example 177. Convert the following equations to polar coordinates:

(a) $x + y = 3$

(b) $y = x^2 + 3x + 1$

Solution. We simply replace $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(a) $r\cos(\theta) + r\sin(\theta) = 3$

(b) $r\sin(\theta) = r^2\cos^2(\theta) + 3r\cos(\theta) + 1$

Example 178. Which shapes are described by the following equations?

(a) $r = 3$

(b) $\theta = \frac{\pi}{4}$

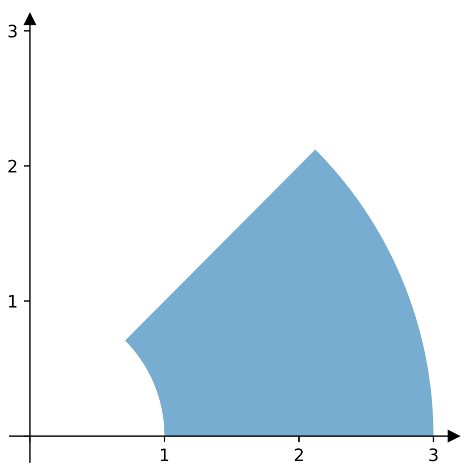
(c) $1 \leqslant r \leqslant 3$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$

Solution.

(a) This is a circle of radius 3 centered at the origin.

(b) This is the line through the origin that is angled $\frac{\pi}{4} = 45°$ up.
(In Cartesian coordinates, this is the line $y = x$.)

(c) The inequality $1 \leqslant r \leqslant 3$ describes an annulus (shaped like a CD: a disk with a hole).
The inequality $0 \leqslant \theta \leqslant \frac{\pi}{4}$ describes a cone.
Putting these two together, the region looks as follows:



Example 179. Describe the y-axis using polar coordinates.

Solution. $\theta = \pm\frac{\pi}{2}$ (just $\theta = \frac{\pi}{2}$ is enough if we also allow $r < 0$).

Alternatively. In Cartesian coordinates, the y-axis is described by the equation $x = 0$. In polar coordinates, this becomes $r\cos(\theta) = 0$. We can simplify this to $r = 0$ (that's just the origin) or $\cos(\theta) = 0$, where the latter becomes $\theta = \pm\frac{\pi}{2}$ (if we work with θ restricted to $(-y = x^2 + 3x + 1\pi, \pi]$).