Example 166. (review) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

The alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$.

For instance, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $(p = 1/2)$ converges. However, it does not converge absolutely.

Series that converge but don't converge absolutely are said to **converge conditionally**.

One has to be more careful with series that only converge conditionally. For instance, we cannot rearrange the order of the terms arbitrarily without affecting the overall sum.

Example 167.

(a) Determine the Taylor series of $f(x) = \cos(x)$ at $x = 0$.

(b) Spell out the Taylor polynomial of order 4 for $f(x) = \cos(x)$ at $x = 0$.

Solution.

(a) The derivatives of $f(x)$ cycle through $\cos(x), -\sin(x), -\cos(x), \sin(x), \ldots$.
In particular, the values $f^{(n)}(0)$ cycle through $1, 0, -1, 0, \ldots$.
That is, $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$.
Therefore, the Taylor series of $f(x) = \cos(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Note. Assuming that $\cos x$ can be written as a power series at $x = 0$, we conclude that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Again, this can be justified via Taylor's formula or a differential equation.

(b) Truncating the Taylor series, the Taylor polynomial of order 4 is

$$1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Example 168. Determine the Taylor series of $\int e^{-x^2} \mathrm{d}x$ at $x = 0$.

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

Integrating term by term, we conclude that $\int e^{-x^2} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C$.

Note. Since e^{-x^2} is an even function, its Taylor series only includes the terms x^{2n} (which are even) and not terms of the form x^{2n+1} (which are odd). See also the Taylor series that we got for $\cos(x)$ (which is even).

Armin Straub
straub@southalabama.edu

Example 169. Determine the Taylor series of $f(x) = x^3 + 3x^2 + 3x + 3$ at $x = -1$.

Solution. By definition, the Taylor series in question is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \ldots$$

Clearly $f(-1) = 2$ and we to compute the other values $f^{(n)}(-1)$ as follows:

- $f'(x) = 3x^2 + 6x + 3$ so that $f'(-1) = 0$.
- $f''(x) = 6x + 6$ so that $f''(-1) = 0$.
- $f''(x) = \frac{3}{8x^{5/2}}$ so that $f'''(1) = \frac{3}{8}$.
- $f'''(x) = 6$ so that $f''(-1) = 6$.
- We note that $f^{(4)}(x) = 0$ so that $f^{(n)}(-1) = 0$ for all $n \geqslant 4$.

The Taylor series therefore is

$$f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 = 2 + \frac{6}{3!}(x+1)^3 = 2 + (x+1)^3.$$

Comment. For a polynomial $f(x)$, the Taylor series is the same polynomial just expanded around a different point. In particular, the Taylor series only has terms up to the degree of $f(x)$.

Example 170. Using familiar simpler power series, find the Taylor series at $x = 0$ for the following:

(a) $\frac{4}{2 + 7x^3}$

(b) $\frac{2}{1 + 3x} + e^{7x}$

Solution.

(a) Using the geometric series, we have $\frac{4}{2+7x^3} = \frac{4}{2} \cdot \frac{1}{1 - \left(-\frac{7}{2}x^3\right)} = 2\sum_{n=0}^{\infty} \left(-\frac{7}{2}x^3\right)^n = 2\sum_{n=0}^{\infty} \left(-\frac{7}{2}\right)^n x^{3n}$.

(b) Using both the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and the exponential series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$\frac{2}{1+3x} + e^{7x} = 2 \cdot \frac{1}{1-(-3x)} + e^{7x} = 2\sum_{n=0}^{\infty} (-3x)^n + \sum_{n=0}^{\infty} \frac{(7x)^n}{n!} = \sum_{n=0}^{\infty} \left[2 \cdot (-3)^n + \frac{7^n}{n!}\right] x^n.$$

Example 171. Find a power series (about $x=0$) for $\frac{1}{1+x^2}$.

Solution. We plug $-x^2$ for x in the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ to get
$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$. This is valid for $|-x^2|<1$ or, equivalently, $|x|<1$.
In particular, this power series has radius of convergence 1.

Example 172. Find a power series (about $x=0$) for $\arctan(x)$.

Solution. Recall that $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x)+C$.
In Example 171, we observed that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ and that this power series converges if $|x|<1$.

We now integrate both sides of $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ to find a power series for $\arctan(x)$.

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$$

Hence, $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$. Since $\arctan(0)=0$, it follows that $C=0$.

Example 173. What is the exact interval of convergence in the previous example?

Solution. Since the convergence radius is 1, we know that the series converges for $|x|<1$, and diverges if $|x|>1$. We don't yet know whether the series converges for $x=\pm 1$.

- For $x=1$, we get the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

  This is an **alternating series** because the terms are alternately positive and negative. Due to the alternating series test, the series converges ($a_n = \frac{1}{2n+1}$ is positive, decreasing and converges to 0).

- For $x=-1$, we get the series $-\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which is -1 times what we get for $x=1$.

  In particular, this series converges as well.

Our conclusion is that the exact interval of convergence is $[-1,1]$.

Comment. The series for $x=\pm 1$ are not **absolutely convergent** because, if we sum instead the absolute values of the terms, then we get $\sum_{n=0}^{\infty} \frac{1}{2n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$, and we know that this series diverges, because it is "half" of the harmonic series. This means that the series for $x=\pm 1$ are only **conditionally convergent**.

Since $\arctan(1) = \frac{\pi}{4}$, we conclude that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.
Note that $\arctan(-1) = -\arctan(1) = -\frac{\pi}{4}$, which explains why we got the same series times -1.