Taylor series

Review. Let $f(x)$ be a function. What is the best linear approximation to $f(x)$ at $x=a$?

Solution. The best linear approximation is the tangent line at $x=a$. This line has slope $f'(a)$ and goes through the point $(a, f(a))$. Hence, the equation for the best linear approximation is $f(a)+f'(a)(x-a)$.

Preview. We will see below that this is the **Taylor polynomial of order** 1. The Taylor polynomials of order M likewise provide the best approximations to $f(x)$ at $x=a$ using polynomials of degree up to M.

Comment. Of course, here we need to restrict to functions $f(x)$ that are differentiable at $x=a$.

The **Taylor series** of $f(x)$ at $x=a$ is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\ldots$$

- The **Taylor polynomial of order** M is the truncation $\sum_{n=0}^{M} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

  This is the best approximation of the function $f(x)$ at $x=a$ using a polynomial of degree up to M.

- If $f(x)$ can be written as a power series $x=a$ (nice functions can be!), then the Taylor series equals $f(x)$.

  Here, we are, of course, restricted to x within the interval of convergence.

- The Taylor series at $x=0$ is also called the **Maclaurin series** of $f(x)$.
- The functions we meet in practice can usually be written as power series (such functions are called **analytic** and are the fundamental object in complex analysis), at least about most points (and it usually is not difficult to tell if a special point is problematic).

  A theoretical guarantee is given by Taylor's formula, which says that

$$f(x)=\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x), \quad \text{with } R_N(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

  for some c between a and x. If $R_N(x)\to 0$ as $N\to\infty$, then $f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Example 162. Determine the Taylor polynomials of order 2 and 3 for $f(x)=\frac{1}{x}$ at $x=3$.

Solution. The Taylor polynomial of order 2 is

$$f(3)+\frac{f'(3)}{1!}(x-3)+\frac{f''(3)}{2!}(x-3)^2=\frac{1}{3}-\frac{1}{9}(x-3)+\frac{1}{27}(x-3)^2.$$

Here, we used that $f'(x)=-\frac{1}{x^2}$ and $f''(x)=\frac{2}{x^3}$ so that $f'(3)=-\frac{1}{9}$ and $f''(3)=\frac{2}{27}$.

For order 3, we also compute $f'''(x)=-\frac{6}{x^4}$ so that $f'''(3)=-\frac{2}{27}$. Hence, the Taylor polynomial of order 3 is

$$f(3)+\frac{f'(3)}{1!}(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3=\frac{1}{3}-\frac{1}{9}(x-3)+\frac{1}{27}(x-3)^2-\frac{1}{81}(x-3)^3.$$

Example 163. Determine the Taylor polynomial of order 3 for $f(x)=\sqrt{x}$ at $x=1$.

Solution. By definition, the Taylor polynomial in question is given by

$$\sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3.$$

Clearly $f(1)=1$ and we to compute the other values $f^{(n)}(1)$ as follows:

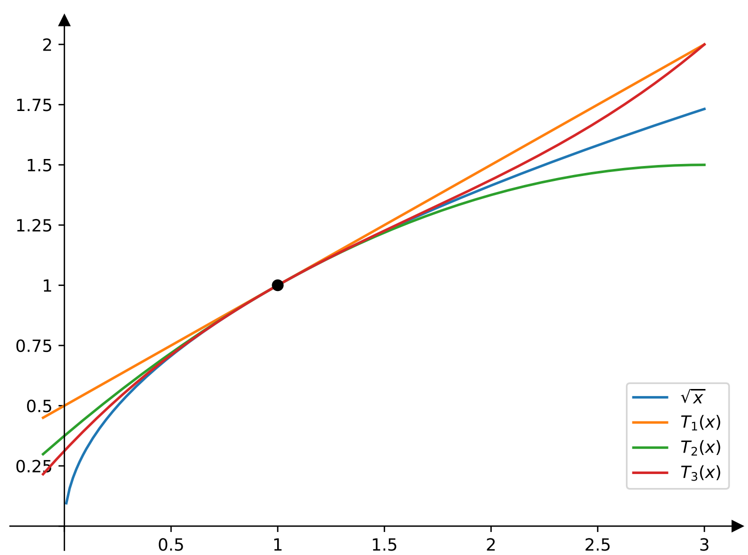
- $f'(x)=\frac{1}{2\sqrt{x}}$ so that $f'(1)=\frac{1}{2}$.
- $f''(x)=-\frac{1}{4x^{3/2}}$ so that $f''(1)=-\frac{1}{4}$.
- $f''(x)=\frac{3}{8x^{5/2}}$ so that $f'''(1)=\frac{3}{8}$.

The Taylor polynomial therefore is

$$f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{6}(x-1)^3=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3.$$

Comment. Note that we can read off any Taylor polynomial $T_M(x)$ of lower order M by just truncating our polynomial. For instance, $T_2(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2$ and $T_1(x)=1+\frac{1}{2}(x-1)$.

The plot below shows how well these Taylor polynomials approximate $f(x)=\sqrt{x}$ near $x=1$.



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Example 164.

(a) Determine the Taylor series of $f(x)=e^x$ at $x=0$.

(b) Spell out the Taylor polynomial of order 4 for $f(x)=e^x$ at $x=0$.

Solution.

(a) All derivatives of $f(x)=e^x$ are $f^{(n)}(x)=e^x$. In particular, $f^{(n)}(0)=1$ for all n.
Therefore, the Taylor series of $f(x)=e^x$ at $x=0$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots$$

Comment. Take the derivative of the power series and observe how it reflects that $\frac{\mathrm{d}}{\mathrm{d}x}e^x=e^x$. Compare with Example 159.

Comment. We have that $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ because e^x is "nice" and can be written as a power series. This can be justified, for instance, using Taylor's formula above.

(b) Truncating the Taylor series, the Taylor polynomial of order 4 is

$$\sum_{n=0}^{4}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}.$$

Example 165. Determine the Taylor series of $f(x)=e^{2x}$ at $x=0$.

Solution. Since $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, it follows that $e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\sum_{n=0}^{\infty}\frac{2^nx^n}{n!}$.

Solution. Observe that $f^{(n)}(x)=2^ne^{2x}$. Hence, $e^{2x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$.

Note. $e^{2x}=e^xe^x=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)=1+2x+2x^2+\frac{4}{3}x^3+\ldots$

[For instance, we get the $\frac{4}{3}x^3$ as $1\cdot\frac{x^3}{6}+x\cdot\frac{x^2}{2}+\frac{x^2}{2}\cdot x+\frac{x^3}{6}\cdot 1=\frac{4}{3}x^3$.]

(Which matches the first terms of our series for e^{2x}.) This illustrates that we can multiply Taylor series.

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