Example 157. What is the radius of convergence of the following power series?

(a) $\sum_{n=0}^{\infty} n!x^n$ (b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution.

(a) We apply the ratio test with $a_n = n!x^n$:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = |x|\,(n+1) \to \infty$ as $n \to \infty$ (unless $|x| = 0$)

The ratio test implies that $\sum_{n=0}^{\infty} n!x^n$ diverges for all x except $x = 0$. The radius of convergence is 0.

(b) We again apply the ratio test, this time with $a_n = \frac{x^n}{n!}$:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!x^{n+1}}{(n+1)!x^n}\right| = \frac{|x|}{n+1} \to 0$ as $n \to \infty$

The ratio test implies that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. The radius of convergence is ∞.

Comment. In Example 159 below, we find that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Power series as functions

The following simply states that we can treat power series as if they were polynomials of infinite degree when it comes to differentiating or integrating.

Theorem 158. (Term-by-term differentiation and integration)

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{on the interval } (a-R, a+R).$$

In this interval, $f(x)$ is arbitrarily often differentiable, and its derivatives can be obtained by differentiating the power series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\, c_n (x-a)^{n-2},$$

and so on. Likewise, $f(x)$ can be integrated term by term:

$$\int f(x)\mathrm{d}x = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

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Example 159. Compute the derivative of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Identify the function $f(x)$.

Solution. We have previously determined that this power series about $x = 0$ has convergence radius $R = \infty$. Therefore, the function $f(x)$ is defined by the series for all x. Its derivative is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

This means that $f(x)$ satisfies the differential equation $y' = y$. It clearly also satisfies $y(0) = 1$. It follows that $f(x)$ is the unique solution of the IVP $y' = y$, $y(0) = 1$. We conclude that $f(x) = e^x$.

Example 160. Differentiate both sides of $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Solution. We find $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$. This identity is valid if $|x| < 1$ because that is the condition under which the geometric series converges.

Comment. If we prefer, we can also write $\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$.

Important observation. The new series $\sum_{n=1}^{\infty} n x^{n-1}$ has again radius of convergence 1 (like the geometric series).

This is a general phenomenon. Differentiating and integrating a power series does not change the radius of convergence. (However, this can change the behaviour at the endpoints of the interval of convergence.)

[Can you see this by thinking about the effect of an additional factor of n in a_n when applying the ratio test?]

Example 161. (extra) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution. The first series is just a geometric series: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1 = 1$

For the second series, we can use $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ with $x = \frac{1}{2}$. In that case, $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$.

Multiplying both sides by $\frac{1}{2}$, we obtain $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

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