Review. Ratio test

The alternating series test

Theorem 149. (Alternating series test) If a_n is positive, decreasing with $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=N}^{\infty} (-1)^n a_n$ converges.

Why? Proof by picture!

Example 150. Does the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

Solution. Yes, it converges by the alternating series test: $a_n = \frac{1}{n}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$.

Important. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ does not **converge absolutely**.

Example 151. For which p does the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge? For which p does it converge absolutely?

Solution.

- If $p > 0$, then the series converges by the alternating series test, because $a_n = \frac{1}{n^p}$ is positive, decreasing, and $\lim_{n\to\infty} a_n = 0$. If $p \leqslant 0$, then $\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ is not zero. Therefore, the series diverges.
- By definition, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Since this is just the usual p-series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely if and only if $p > 1$.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges if and only if $p > 0$, and converges absolutely if and only if $p > 1$.

Power series

Definition 152. A **power series** (about $x = 0$) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

More generally, a **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

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Example 153. Determine all x for which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

Solution. We apply the ratio test with $a_n = \frac{x^n}{n}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x|$ as $n \to \infty$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if $|x| < 1$ (and diverges if $|x| > 1$).

Note that the ratio test does not tell us what happens when $|x| = 1$. We need to look at those cases more carefully:

- $x = 1$: In this case we get the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know diverges.
- $x = -1$: In this case we get the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which we know converges.

In summary, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if and only if x is in $[-1, 1)$.

Note. This is a power series around 0. The series converges for all x less than 1 away from 0. This is why we say that the power series has **radius of convergence** 1.

Looking ahead. Which function is hiding behind $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$?

Power series can be differentiated term-by-term and so $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Therefore, $f(x) = \int \frac{1}{1-x}\mathrm{d}x = -\ln|1-x| + C$.

Finally, the fact that $f(0) = 0$ implies that $C = 0$. This means that $f(x) = -\ln|1-x|$.

Note that this function is nice at $x = 0$ (and around it) but that it has a problem for $x = 1$. Since converging power series do not have problems, this is one way to see that the radius of convergence is 1.

Let us again look at the special values $x = 1$ and $x = -1$. First, we clearly have $\lim_{x \to 1} f(x) = +\infty$.

On the other hand, $\lim_{x \to -1} f(x) = -\ln(2)$ implying that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots = -\ln(2) \approx -0.693$.

Theorem 154. Every power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a **radius of convergence** R, meaning:

(a) if $R = 0$, then the series converges only for $x = a$,

(b) if $0 < R < \infty$, then the series converges for all x such that $|x-a| < R$ but diverges if $|x-a| > R$ (in other words, R is as large as possible),

(c) if $R = \infty$, then the series converges for all x.

Note that, if $0 < R < \infty$, no general statement can be made for the case $|x-a| = R$.

The exact **interval of convergence** can be $(a-R, a+R)$ or $[a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R]$.