Notes for Lecture 30

Wed, 11/6/2024

Review. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Example 145. Determine whether the following series converge or diverge.

(a) $\sum_{n=0}^{\infty} \frac{1}{2n+1}$

(b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

(c) $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$

(d) $\sum_{n=1}^{\infty} \frac{2\sqrt{n}+1}{3n^2+3}$

Solution.

(a) We could do an integral comparison test (do it!) or we could do a direct comparison (do it!) but, when possible, it is easiest to do a limit comparison test. Namely, we can "see" that the terms $a_n = \frac{1}{2n+1}$ behave like $\frac{1}{2n}$ for large n.
Therefore, we do a limit comparison with $b_n = \frac{1}{n}$ (you could also choose $b_n = \frac{1}{2n}$ but that factor of 2 is not relevant):

$$\frac{a_n}{b_n} = \frac{n}{2n+1} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Since the limit is not zero and finite, limit comparison tells us that $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic, we know that it diverges. It follows that our series diverges as well.

(b) We proceed as in the previous part but now do a limit comparison with $b_n = \frac{1}{n^2}$ to conclude that our series converges (note that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p > 1$ and so converges).

(c) We do a limit comparison of the sequence $a_n = \frac{2n+1}{3n^2+3}$ with $b_n = \frac{1}{n}$.
First, we check that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3}$. By the limit comparison, we then find that $\sum_{n=1}^{\infty} \frac{2n+1}{3n^2+3}$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(d) We do a limit comparison with $b_n = \frac{1}{n^{3/2}}$ to conclude that both series converge.

The ratio test

(absolute convergence)

We say that the series $\sum_{n=N}^{\infty} a_n$ **converges absolutely** if $\sum_{n=N}^{\infty} |a_n|$ converges.

It is not hard to see (check out Section 9.5 in the book) that absolute convergence implies (regular) convergence.
It is often easier to work with series where all terms are $\geqslant 0$. Therefore, it is often easier to establish absolute convergence of a series (and then get convergence for free). Some tests like the **ratio test** even give us absolute convergence.

Caution! There are series which converge but which do not converge absolutely.

One example is the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$. We will discuss **alternating series** soon.

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Note that a series $\sum_{n=N}^{\infty} a_n$ is geometric if $\frac{a_{n+1}}{a_n} = L$ is constant. It converges if and only if $|L| < 1$.

Theorem 146. (Ratio test) Suppose the limit $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists.

- If $L < 1$, then the series $\sum_{n=N}^{\infty} a_n$ converges (absolutely).
- If $L > 1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges.
- If $L = 1$, then we don't know. The test is inconclusive.

Example 147. Apply the ratio test to the geometric series $\sum_{n=0}^{\infty} x^n$.

Solution. In this case, $a_n = x^n$ and so $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{x^n} \right| = |x|$.

The ratio test with $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|$ then shows that $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$, and diverges if $|x| > 1$.

Important. The ratio test makes no statement about the cases $x = 1$ and $x = -1$. In these cases, we need to do additional analysis. Here, it is easy to see directly that the geometric series diverges when $x = 1$ or $x = -1$.

Example 148. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$

Solution.

(a) We apply the ratio test with $a_n = \frac{n^2}{2^n}$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2} = \frac{1}{2} \frac{(n+1)^2}{n^2} = \frac{1}{2} \frac{n^2 + 2n + 1}{n^2} \to \frac{1}{2} \text{ as } n \to \infty$$

Since $\frac{1}{2} < 1$, the ratio test implies that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(b) Note that $\frac{2^n}{n^3} \to \infty \neq 0$. Hence, the series diverges.

Alternatively. Suppose we didn't realize this and, instead, we apply the ratio test with $a_n = \frac{2^n}{n^3}$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)^3} \frac{n^3}{2^n} = 2 \frac{n^3}{(n+1)^3} = 2 \frac{n^3}{n^3 + 3n^2 + 3n + 1} \to 2 \text{ as } n \to \infty$$

Since $2 > 1$, the ratio test implies that $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges.

(c) We apply the ratio test with $a_n = \frac{(-1)^n 5^n}{n!}$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} = \frac{5}{n+1} \to 0 \text{ as } n \to \infty$$

Since $0 < 1$, the ratio test implies that $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{n!}$ converges.

Review. Recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. This is the **factorial**.
It counts the number of ways in which you can order n objects.