Integral comparison test

Theorem 138. (Integral comparison test) Suppose that the function $f(x)$ is positive, continuous and decreasing for $x \geqslant N$. Then:

$$\sum_{n=N}^{\infty} f(n) \text{ converges } \iff \int_N^{\infty} f(x)\mathrm{d}x \text{ converges}$$

In other words, the series and integral both converge or both diverge.

Why? Make a sketch where you compare the area under the curve $f(x)$ with rectangles of width 1. (See Section 9.3 in our book for nice illustrations.)

Warning: if they converge, of course, the values of the series and the integral are going to be different!

Example 139. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges. Why?

Solution. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$, so we cannot directly use our test for divergence coming out of Theorem 136. However, we can combine terms as follows to see the divergence:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geqslant \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geqslant \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geqslant \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}} + \ldots$$

Solution. (integral comparison test) The function $f(x) = \frac{1}{x}$ is positive, continuous and decreasing for $x \geqslant 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ converges } \iff \int_1^{\infty} \frac{1}{x}\mathrm{d}x \text{ converges.}$$

Since $\int_1^M \frac{1}{x}\mathrm{d}x = \left[\ln|x|\right]_1^M = \ln M \to \infty$ as $M \to \infty$, the integral diverges. It follows, by comparison, that the harmonic series diverges, too.

Example 140. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converges.

[It is considerably more difficult to show that, in fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

Solution. As in the previous example, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if and only if the integral $\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x$ converges. Since $\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x = \left[-\frac{1}{x}\right]_1^{\infty} = 0 - (-1) = 1$, the integral converges, and so the series converges as well.

More generally, we have the following result:

(p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a **p-series**. It converges if and only if $p > 1$.

Why? This follows from the integral comparison test, and because $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$. See next example.

Armin Straub
straub@southalabama.edu

Example 141. For what values of p does $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ converge?

Solution. For $p \neq 1$, we have $\int_1^\infty \frac{\mathrm{d}x}{x^p} = \left[\frac{1}{-p+1} x^{-p+1}\right]_1^\infty$.

If $p > 1$, then $\lim_{x\to\infty} x^{-p+1} = \lim_{x\to\infty} \frac{1}{x^{p-1}} = 0$, and we find that the integral converges.

If $p < 1$, then $\lim_{x\to\infty} x^{-p+1} = \infty$, and we find that the integral diverges.

We are missing only the case $p = 1$: in that case, $\int_1^\infty \frac{1}{x}\mathrm{d}x = \left[\ln|x|\right]_1^\infty$ diverges because $\lim_{x\to\infty} \ln|x| = \infty$.

In summary, $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$.

Example 142. Determine whether the following series converge or diverge.

(a) $\sum_{n=0}^\infty \frac{1}{2n+1}$

(b) $\sum_{n=0}^\infty \frac{1}{n^2+1}$

Solution. You can use the integral comparison test to find that the first series diverges and the second series converges. However, it is more convenient to look at these series by comparison which is what we discuss next.

Direct comparison and limit comparison

Recall that we discussed direct comparison and limit comparison tests for improper integrals. The same ideas apply to series:

We assume that both $a_n \geqslant 0$ and $b_n \geqslant 0$.

(direct comparison)

- $\sum_{n=N}^\infty a_n$ converges if $a_n \leqslant b_n$ and $\sum_{n=N}^\infty b_n$ converges.
- $\sum_{n=N}^\infty a_n$ diverges if $a_n \geqslant b_n$ and $\sum_{n=N}^\infty b_n$ diverges.

(limit comparison)

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for $0 < L < \infty$ then $\sum_{n=N}^\infty a_n$ and $\sum_{n=N}^\infty b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=N}^\infty b_n$ converges, then $\sum_{n=N}^\infty a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=N}^\infty b_n$ diverges, then $\sum_{n=N}^\infty a_n$ diverges.

Armin Straub
straub@southalabama.edu

Example 143. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+\ln(n)}$ (b) $\sum_{n=1}^{\infty} \frac{n+\ln(n)}{n^2+4}$

Solution.

(a) **(direct comparison)** Note that, for all $n \geqslant 1$, $n^2+\log(n) \geqslant n^2$ and so $\frac{1}{n^2+\log(n)} \leqslant \frac{1}{n^2}$.

By comparison, $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} = \text{finite}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)}$ converges.

(limit comparison) Observe that we can just "see" this: for large n, our terms $\frac{1}{n^2+\log(n)}$ "behave" like $\frac{1}{n^2}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2+\log(n)}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This reasoning is made precise by the limit comparison test. Spell out the details!

Comment. Like in this example, we often have a "natural" comparison with a geometric series or a p-series. In those cases, don't spend time thinking about the corresponding integral! Here, $\int_1^{\infty} \frac{dx}{x^2+\log(x)}$ is such that its antiderivative cannot even be written in terms of the functions we are familiar with.

(b) Our terms $a_n = \frac{n+\ln(n)}{n^2+4}$ behave like $\frac{n}{n^2} = \frac{1}{n}$ for large n. Thus, we should do limit comparison with $b_n = \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n+\ln(n))}{n^2+4} = 1.$$

This means that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (this is the harmonic series), our series diverges as well.

Frequently, we have choices which test to apply to determine whether a series converges:

Example 144. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. (via integral comparison) The integral $\int_1^{\infty} \frac{\ln x}{x} dx = \int_0^{\infty} u\, du$ obviously diverges (note that we substituted $u = \ln x$), and hence $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges as well.

Comment. Usually, it is a good idea to avoid integral comparison unless there are no easier options.

Solution. (direct comparison) Note that $\frac{\log n}{n} > \frac{1}{n}$ for all $n > 3$ (because $\log n > 1$ for $n > 3$).

But already $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (note that $\sum_{n=3}^{\infty} \frac{\log n}{n} > \sum_{n=3}^{\infty} \frac{1}{n}$), so $\sum_{n=1}^{\infty} \frac{\log n}{n}$ has to diverge as well.

Solution. (limit comparison) We do limit comparison of $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$ so that $\frac{a_n}{b_n} = \ln n$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, we conclude that $\sum_{n=N}^{\infty} a_n$ diverges as well.