Sequences

A **sequence**, often denoted $\{a_n\}$, is an infinite list of its **terms** $a_1, a_2, a_3, \ldots$.
We'll define it precisely later, but one thing we are interested in is the **limit** $\lim_{n\to\infty} a_n$ (if it exists).
Here are a few first examples of sequences:

- $2,\ 4,\ 6,\ 8,\ 10,\ \ldots$ (that is, $a_1 = 2$, $a_2 = 4$, ...)

  This is the sequence $\{a_n\}$ with $a_n = 2n$. Clearly, $\lim_{n\to\infty} a_n = \infty$.

- $1,\ -1,\ 1,\ -1,\ 1,\ \ldots$

  This is the sequence $\{a_n\}$ with $a_n = (-1)^{n-1}$. The limit $\lim_{n\to\infty} a_n$ does not exist.

  Comment. Some part of the sequence "goes to" 1 but another part to -1. There is no single value that all terms approach.

- $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$

  This is the sequence $\{a_n\}$ with $a_n = \frac{1}{2^n}$. Clearly, $\lim_{n\to\infty} a_n = 0$.

  Preview. We will learn later that the **series** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$ also converges and equals 1. Can you maybe already explain why this is the case?

- $3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ \ldots$

  This is the sequence $\{a_n\}$ where a_n consists of the first n (decimal) digits of π. Clearly, $\lim_{n\to\infty} a_n = \pi$.

- $\frac{1}{1},\ \frac{1}{4},\ \frac{1}{9},\ \frac{1}{16},\ \frac{1}{25},\ \ldots$

  This is the sequence $\{a_n\}$ with $a_n = \frac{1}{n^2}$. Clearly, $\lim_{n\to\infty} a_n = 0$.

  Preview. We will learn later that the **series** $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$ also converges and equals $\frac{\pi^2}{6}$.

- $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$

  These are the **Fibonacci numbers** $\{F_n\}$. They are defined **recursively**: $F_n = F_{n-1} + F_{n-2}$ together with the **initial values** $F_1 = 1$, $F_2 = 1$. Clearly, $\lim_{n\to\infty} F_n = \infty$.

- $\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{21}{13},\ \frac{34}{21},\ \ldots$

  These are quotients of Fibonacci numbers $\{a_n\}$ with $a_n = \frac{F_{n+1}}{F_n}$.

  Numerically, $1,\ 2,\ 1.5,\ 1.667,\ 1.6,\ 1.625,\ 1.615,\ 1.619,\ \ldots$ Looks like $\lim_{n\to\infty} a_n$ exists and is about 1.618.

Sequences and series. In a little bit, we will also be interested in **series**. These are infinite sums such as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$. Do not confuse these two!

Confusing sequences and series would be like confusing a function and its definite integral.

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Limits of sequences

If $\lim_{x\to\infty} f(x) = L$ (the limit of a function: x is real)

then $\lim_{n\to\infty} f(n) = L$ (the limit of a sequence: n is an integer).

Important. The reverse is not true: for instance, $\lim_{n\to\infty} \sin(\pi n) = 0$ but $\lim_{x\to\infty} \sin(\pi x)$ does not exist.

Example 122. Determine the following limits:

(a) $\lim_{n\to\infty} \dfrac{3n^2+7n-8}{8n^2+n+1}$

(b) $\lim_{n\to\infty} \dfrac{3n^2+7n-8}{8n^3+n+1}$

(c) $\lim_{n\to\infty} \dfrac{e^{3n}}{n^2}$

(d) $\lim_{n\to\infty} \dfrac{\sin(n)}{n}$

(e) $\lim_{n\to\infty} \cos\left(\pi - \frac{1}{n^2}\right)$

Solution. In each example, try to first "see" the limit! Then, apply some technique (like L'Hospital) to confirm.

(a) $\lim_{n\to\infty} \dfrac{3n^2+7n-8}{8n^2+n+1} = \dfrac{3}{8}$

The easiest way to see this is to note that the main terms in the numerator and denominator are $3n^2$ and $8n^2$. It follows that the limit is the same as $\lim_{n\to\infty} \frac{3n^2}{8n^2} = \lim_{n\to\infty} \frac{3}{8} = 0$.

We can make this argument precise in two different ways:

- $\dfrac{3n^2+7n-8}{8n^2+n+1} = \dfrac{3+\frac{7}{n}-\frac{8}{n^2}}{8+\frac{1}{n}+\frac{1}{n^2}}$ and now we can observe that all terms like $7/n$ go to 0 as $n\to\infty$.
- Since the quotient is of the undetermined form "$\frac{\infty}{\infty}$", we can apply L'Hospital (twice):

$$\lim_{n\to\infty} \frac{3n^2+7n-8}{8n^2+n+1} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{6n+7}{16n+1} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{6}{16} = \frac{3}{8}$$

(b) $\lim_{n\to\infty} \dfrac{3n^2+7n-8}{8n^3+n+1} = 0$

(c) $\lim_{n\to\infty} \dfrac{e^{3n}}{n^2} = \infty$

This is clear if you keep in mind that exponential growth exceeds any polynomial growth.

If needed, we can apply L'Hospital (twice!) since the limit is of the form "$\frac{\infty}{\infty}$":

$$\lim_{n\to\infty} \frac{e^{3n}}{n^2} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{3e^{3n}}{2n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{9e^{3n}}{2} = \infty$$

(d) $\lim_{n\to\infty} \dfrac{\sin(n)}{n} = 0$

Note that $-\frac{1}{n} \leqslant \frac{\sin(n)}{n} \leqslant \frac{1}{n}$. Since our sequence is squeezed between two sequences which approach 0, our limit has to be 0 as well.

Important. We cannot apply L'Hospital because the limit is not of the form "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$". If we did, we would get the limit $\lim_{n\to\infty} \frac{\cos(n)}{1}$ which does not exist (because the values oscillate between -1 and 1).

(e) $\lim_{n\to\infty} \cos\left(\pi - \dfrac{1}{n^2}\right) = \cos(\pi) = -1$

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