Review. Direct comparison and limit comparison

L'Hospital's rule

Example 119. The following example illustrates that limits of the form "$\frac{\infty}{\infty}$" are completely undetermined. Anything is possible for the actual limit:

- $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$
- $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to\infty} \frac{x}{3x} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}$
- $\lim_{x\to\infty} \frac{x(1+\sin^2(x))}{x} = \lim_{x\to\infty} (1+\sin^2(x))$ This limit does not exist.

Theorem 120. (L'Hospital's rule) If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

The same conclusion holds if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

[It is important to realize that L'Hospital's rule only applies to the undetermined cases "$\frac{\infty}{\infty}$" and "$\frac{0}{0}$".]

Example 121. $\int_0^\infty x e^{-3x} \mathrm{d}x =$

Your final answer should be $\frac{1}{9}$.

Along the way, you will need the limit $\lim_{x\to\infty} x e^{-3x} = \lim_{x\to\infty} \frac{x}{e^{3x}} \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{3e^{3x}} = 0$.

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