Example 114. Determine $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x$.

Make a sketch! (For that, note that the integrand is always positive and that it goes to 0 as $x \to \infty$ or $x \to -\infty$. Also, you can see that the maximum occurs at $x = 0$. Taken together, the graph looks like a single mound.)

Your final answer should be π.

Solution. Recall that $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan(x) + C$. If necessary, review $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and its inverse function $\arctan(x)$ to recall that

$$\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}, \qquad \lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}.$$

We therefore have $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x = \left[\arctan(x)\right]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

The following integral is an example of an improper integral of type II (because the integrand has a vertical asymptote at one of the limits).

Example 115. Determine $\int_0^1 \frac{1}{x} \mathrm{d}x$.

Make a sketch!

Solution. $\int_0^1 \frac{1}{x} \mathrm{d}x = \left[\ln|x|\right]_0^1$ but $\lim_{x \to 0^+} \ln|x| = -\infty$.

Thus, the integral diverges (to ∞, in this case).

Example 116. The following is VERY WRONG:

$$[\text{bad!}] \qquad \int_{-2}^{2} \frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_{-2}^{2} = -\frac{1}{3} - 1 = -\frac{4}{3} \qquad [\text{bad!}]$$

Note how even the answer is screaming trouble: we integrated something positive and got a negative value.

What went wrong? The integrand has a problem at $x = -1$!

To be precise, it has a vertical asymptote at $x = -1$. Since $\frac{1}{(x+1)^2}$ is not differentiable (not even continuous) at $x = -1$, we can only use the antiderivative $-\frac{1}{x+1}$ for $x \neq -1$. Since -1 is in the domain of integration $[-2, 2]$, we cannot directly apply the Fundamental Theorem of Calculus to this integral.

Instead, we need to split the integral into two improper integrals and analyze these individually:

$$\int_{-2}^{2} \frac{\mathrm{d}x}{(x+1)^2} = \int_{-2}^{-1} \frac{\mathrm{d}x}{(x+1)^2} + \int_{-1}^{2} \frac{\mathrm{d}x}{(x+1)^2}.$$

But $\int_{-2}^{-1} \frac{\mathrm{d}x}{(x+1)^2} = \left[-\frac{1}{x+1}\right]_{-2}^{-1}$ diverges because $\lim_{x \to -1^-} \left(-\frac{1}{x+1}\right) = \infty$. Hence, our original integral diverges.

[Note that, working on the second integral, the limit we encounter is $\lim_{x \to -1^+} \left(-\frac{1}{x+1}\right) = -\infty$. This integral, by itself, also diverges.]

Example 117. Does $\int_2^3 \frac{\mathrm{d}x}{x-2}$ converge? Does $\int_2^3 \frac{\mathrm{d}x}{\sqrt{x-2}}$ converge?

Solution. Final answers: No. Yes.

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Direct comparison and limit comparison

Sometimes we just want to know if an integral converges or diverges. In that case, we can compare the integrand with simpler functions.

The following illustrates our approach but is not meant to be exhaustive. The same ideas apply (suitably adjusted), for instance, when the functions are negative.

We assume that both $f(x)$ and $g(x)$ are continuous on $[a,\infty)$ and positive.

(direct comparison)

- $\int_a^\infty f(x)\mathrm{d}x$ converges if $0 \leqslant f(x) \leqslant g(x)$ on $[a,\infty)$ and $\int_a^\infty g(x)\mathrm{d}x$ converges.
- $\int_a^\infty f(x)\mathrm{d}x$ diverges if $f(x) \geqslant g(x) \geqslant 0$ on $[a,\infty)$ and $\int_a^\infty g(x)\mathrm{d}x$ diverges.

(limit comparison)

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ for $0 < L < \infty$ then
  $\int_a^\infty f(x)\mathrm{d}x$ converges if and only if $\int_a^\infty g(x)\mathrm{d}x$ converges.

Example 118. Determine whether the following integrals converge or diverge.

(a) $\int_1^\infty \frac{1}{x^5+2}\mathrm{d}x$

(b) $\int_2^\infty \frac{\sqrt{x^2+4}}{x^2}\mathrm{d}x$

(c) $\int_2^\infty \frac{\sqrt{x+4}}{x^2}\mathrm{d}x$

(d) $\int_2^\infty \frac{4-\sin(x)}{x^2}\mathrm{d}x$

(e) $\int_2^\infty \frac{4-\sin(x)}{x}\mathrm{d}x$

(f) $\int_1^\infty \frac{e^x}{x^2}\mathrm{d}x$

(g) $\int_1^\infty \frac{1}{\sqrt{e^{2x}+3x}}\mathrm{d}x$

Solution. The following are just indications of how to proceed. Fill in the details!

(a) We can apply the limit comparison test with $\frac{1}{x^5+2}$ and $\frac{1}{x^5}$ because $\lim_{x\to\infty} \frac{1/(x^5+2)}{1/x^5} = 1$.

Since $\int_1^\infty \frac{1}{x^5}\mathrm{d}x = \left[-\frac{1}{4x^4}\right]_1^\infty = \frac{1}{4}$ converges, it follows that $\int_1^\infty \frac{1}{x^5+2}\mathrm{d}x$ converges as well.

(b) Do limit comparison with $\frac{\sqrt{x^2}}{x^2} = \frac{1}{x}$ to conclude that this integral diverges.

(c) Do limit comparison with $\frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$ to conclude that this integral converges.

(d) Do a direct comparison with $\frac{5}{x^2}$ to conclude that this integral converges.

(e) Do a direct comparison with $\frac{3}{x}$ to conclude that this integral diverges.

(f) Note that $\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$. Hence the integral obviously diverges.

(g) Do limit comparison with $\frac{1}{\sqrt{e^{2x}}} = e^{-x}$ to conclude that this integral converges.

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