Notes for Lecture 22

Wed, 10/16/2024

Example 111. Evaluate $\int \frac{2x+1}{x^2+6x+9}\mathrm{d}x$.

- This time, after factoring, partial fractions tells us that

$$\frac{2x+1}{x^2+6x+9} = \frac{2x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}.$$

- Clearing denominators, $2x+1 = (x+3)A + B$. Setting $x = -3$, we find $-5 = B$. There is no "magic" next choice for x so we just set $x = 0$ (any other choice works as well) to get $1 = 3A - 5$, which implies $A = 2$.

- Integration now is again straightforward (make sure it is to you!):

$$\int \frac{2x+1}{x^2+6x+9}\mathrm{d}x = \int \frac{2}{x+3}\mathrm{d}x + \int \frac{-5}{(x+3)^2}\mathrm{d}x = 2\ln|x+3| + \frac{5}{x+3} + C.$$

Comment. While there is no "magic" next choice for x, we can take derivatives to get rid of B! Indeed, differentiating $2x+1 = (x+3)A + B$ gives $2 = A$ directly.

Comment. The form $\frac{A}{x+3} + \frac{B}{(x+3)^2}$ is equivalent to the form $\frac{Cx+D}{(x+3)^2}$. However, the former is more useful for integrating.

Improper integrals

Example 112. Determine $\int_0^\infty e^{-x}\mathrm{d}x$.

This integral is an example of an **improper integral** of type I (because one of its limits is ∞).

Make a sketch!

Solution. Replacing the upper limit with b, we have $\int_0^b e^{-x}\,\mathrm{d}x = \left[-e^{-x}\right]_0^b = 1 - e^{-b}$.

Therefore, $\int_0^\infty e^{-x}\mathrm{d}x = \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}(1 - e^{-b}) = 1$.

Solution. (short version) Once experienced, we can just write $\int_0^\infty e^{-x}\mathrm{d}x = \left[-e^{-x}\right]_0^\infty = 0 - (-1) = 1$ with the understanding that we used $\lim_{x\to\infty} -e^{-x} = 0$.

Example 113. Determine $\int_1^\infty \frac{1}{x^4}\mathrm{d}x$ as well as $\int_1^\infty \frac{1}{x}\mathrm{d}x$.

Make a sketch! In the first quadrant, both functions look pretty similar.

Solution.

(a) $\int_1^\infty \frac{1}{x^4}\mathrm{d}x = \left[-\frac{1}{3x^3}\right]_1^\infty = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$ where we used that $\lim_{x\to\infty} -\frac{1}{3x^3} = 0$.

(b) $\int_1^\infty \frac{1}{x}\mathrm{d}x = \left[\ln|x|\right]_1^\infty$ but $\lim_{x\to\infty}\ln|x| = \infty$. We thus say that this integral **diverges** (to ∞ in this case).