Partial fractions

Review. rational function $= \dfrac{\text{polynomial}}{\text{another polynomial}}$

Example 103. We are surely all familiar with putting stuff on a common demoninator like in

$$\frac{2}{x+1} + \frac{3}{x-1} = \frac{2(x-1)+3(x+1)}{(x+1)(x-1)} = \frac{5x+1}{(x+1)(x-1)}.$$

Partial fractions refers to reversing this process of putting things on a common denominator.

Example 104. The previous example allows us to easily compute the following integral:

$$\int \frac{5x+1}{(x+1)(x-1)} \mathrm{d}x = \int \frac{2}{x+1} \mathrm{d}x + \int \frac{3}{x-1} \mathrm{d}x = 2\ln|x+1| + 3\ln|x-1| + C.$$

Make sure that you are comfortable with integrating the two simpler integrals!
[For instance, notice that substituting $u = x+1$ in the first of the two, we have $\mathrm{d}u = \mathrm{d}x$ which is why we just get $\log$ of $x+1$.]

Example 105. Evaluate $\displaystyle\int \frac{x+4}{x(x-2)} \mathrm{d}x$ by partial fractions.

Solution. Partial fractions tells us that $\frac{x+4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$ for some numbers A, B that we still need to find:

- To find A and B we multiply both sides with $x(x-2)$ to clear denominators:

$$x+4 = (x-2)A + xB$$

- This equation has to be true for all values of x:
  Set $x=0$ to get $4 = -2A$ so that $A = -2$.
  Set $x=2$ to get $6 = 2B$ so that $B = 3$.

- We can now verify that, indeed, $\frac{x+4}{x(x-2)} = \frac{-2}{x} + \frac{3}{x-2}$.

We therefore have $\displaystyle\int \frac{x+4}{x(x-2)} \mathrm{d}x = \int \frac{-2}{x} \mathrm{d}x + \int \frac{3}{x-2} \mathrm{d}x = -2\ln|x| + 3\ln|x-2| + C.$

Important. Setting $x=0$ and $x=2$ makes our life particularly easy. On the other hand, note that we can set x to any values to get valid equations for A and B (once we have two equations for those two unknowns, we should be able to solve for them).
Alternatively, note that both sides of $x+4 = (x-2)A + xB$ are polynomials in x. We can therefore also equate coefficients: comparing the coefficients of x gives $1 = A+B$ while comparing constant coefficients gives $4 = -2A$. Solving these, we again find $A = -2$ and $B = 3$.

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Example 106. Evaluate $\int \frac{2x-5}{x(x+1)(x+2)}\mathrm{d}x$ by partial fractions.

Solution. Partial fractions tells us that $\frac{2x-5}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$ for certain numbers A, B, C.

- We multiply both sides with $x(x+1)(x+2)$ to clear denominators:

$$2x-5 = (x+1)(x+2)A + x(x+2)B + x(x+1)C$$

- Set $x=0$ to get $-5 = 2A$ so that $A = -\frac{5}{2}$.
  Set $x=-1$ to get $-7 = -B$ so that $B = 7$.
  Set $x=-2$ to get $-9 = 2C$ so that $C = -\frac{9}{2}$.

- Consequently: $\frac{2x-5}{x(x+1)(x+2)} = -\frac{5}{2}\cdot\frac{1}{x} + \frac{7}{x+1} - \frac{9}{2}\cdot\frac{1}{x+2}$.

Hence: $\int \frac{2x-5}{x(x+1)(x+2)}\mathrm{d}x = -\frac{5}{2}\int\frac{\mathrm{d}x}{x} + 7\int\frac{\mathrm{d}x}{x+1} - \frac{9}{2}\int\frac{\mathrm{d}x}{x+2} = -\frac{5}{2}\ln|x| + 7\ln|x+1| - \frac{9}{2}\ln|x+2| + C.$

To decompose the rational function $\frac{f(x)}{g(x)}$ into partial fractions:

(a) Check that $\text{degree } f(x) < \text{degree } g(x)$. (Otherwise, long division!)

(b) Factor $g(x)$ as far as possible.

(c) For each factor of $g(x)$ collect terms as follows:

- For a linear factor $x-r$, occuring as $(x-r)^m$ in $g(x)$, these terms are

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \ldots + \frac{A_m}{(x-r)^m}.$$

- For a quadratic factor x^2+px+q, occuring as $(x^2+px+q)^m$ in $g(x)$, these terms are

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \ldots + \frac{B_mx+C_m}{(x^2+px+q)^m}.$$

(d) Determine the values of the unknown constants (the A's, B's and C's).

Example 107. Evaluate $\int \frac{x^4+3x^3+1}{x^3-x}\mathrm{d}x$.

Solution. (outline) In order to proceed as in the previous problem, we need to address two things:

(a) Since the degree of the numerator is not less than the degree of the denominator, we need to first perform long division. In this case, we get

$$\frac{x^4+3x^3+1}{x^3-x} = x+3+\frac{x^2+3x+1}{x^3-x}.$$

(b) Factor the denominator: $x^3-x = x(x^2-1) = x(x-1)(x+1)$.

Therefore,

$$\int \frac{x^4+3x^3+1}{x^3-x}\mathrm{d}x = \int\left(x+3+\frac{x^2+3x+1}{x(x-1)(x+1)}\right)\mathrm{d}x = \frac{1}{2}x^2+3x+\int\frac{x^2+3x+1}{x(x-1)(x+1)}\mathrm{d}x.$$

We then can evaluate the final integral as in the previous problem.