One exponent may also be negative (in the next example, we integrate $[\sin(x)]^1\,[\cos(x)]^{-1}$).

Example 96. Determine $\int \tan(x)\,\mathrm{d}x$.

Solution. $\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\mathrm{d}x$, so we substitute $u = \cos(x)$ (then $\mathrm{d}u = -\sin(x)\mathrm{d}x$) to get

$$\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\mathrm{d}x = -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

Solution. (harder—only for practice) For some exercise in substituting, we can also substitute $u = \sin(x)$ (but can you explain how we can tell beforehand that $u = \cos(x)$ should be the better choice?). Then $\mathrm{d}u = \cos(x)\mathrm{d}x$ or, equivalently, $\mathrm{d}x = \frac{1}{\cos(x)}\mathrm{d}u$, so that we get

$$\int \tan(x)\,\mathrm{d}x = \int \frac{\sin(x)}{\cos(x)}\mathrm{d}x = \int \frac{u}{\cos^2(x)}\mathrm{d}u = \int \frac{u}{1-\sin^2(x)}\mathrm{d}u = \int \frac{u}{1-u^2}\mathrm{d}u.$$

We now substitute $v = 1 - u^2$ (so that $\mathrm{d}v = -2u\mathrm{d}u$) to get

$$\begin{aligned}\int \tan(x)\,\mathrm{d}x &= \int \frac{u}{1-u^2}\mathrm{d}u = -\frac{1}{2}\int \frac{\mathrm{d}v}{v} = -\frac{1}{2}\ln|v| + C = -\frac{1}{2}\ln|1-u^2| + C\\ &= -\frac{1}{2}\ln|1-\sin^2(x)| + C = -\frac{1}{2}\ln|\cos^2(x)| + C = -\ln|\cos(x)| + C\end{aligned}$$

as earlier.

Trigonometric substitutions

Example 97. Everybody knows that $\cos^2 x + \sin^2 x = 1$.
Divide both sides by $\cos^2 x$ to find $1 + \tan^2 x = \sec^2 x$.

Likewise, dividing by $\sin^2 x$, we find $\cot^2 x + 1 = \csc^2(x)$. However, note that in this identity we cannot have $x = 0$.

if you see	**try substituting**	because
$a^2 - x^2$ (especially $\sqrt{a^2 - x^2}$)	$x = a\sin\theta$	$a^2 - (a\sin\theta)^2 = a^2\cos^2\theta$
$a^2 + x^2$ (especially $\sqrt{a^2 + x^2}$)	$x = a\tan\theta$	$a^2 + (a\tan\theta)^2 = a^2\sec^2\theta = \frac{a^2}{\cos^2\theta}$
and, somewhat less importantly:		
$x^2 - a^2$ (especially $\sqrt{x^2 - a^2}$)	$x = a\sec\theta$	$(a\sec\theta)^2 - a^2 = a^2\tan^2\theta$

Note that (by completing the square and doing a simple linear substitution), you can put any quadratic term $ax^2 + bx + c$ into one of these three cases (for instance, $x^2 + 2x + 3 = (x+1)^2 + 2 = u^2 + 2$ with the simple linear substitution $u = x + 1$).
This is why trigonometric substitution occurs frequently for certain kinds of integrals.

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Example 98. Determine $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$.

Solution. We substitute $x = \sin\theta$ (with $\theta \in [-\pi/2, \pi/2]$ so that $\theta = \arcsin(x)$) because then $1 - x^2 = \cos^2\theta$. Since $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$, we find

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{\cos^2\theta}} = \int 1\,\mathrm{d}\theta = \theta + C = \arcsin(x) + C.$$

[Note that in order to conclude $\sqrt{\cos^2\theta} = \cos\theta$, we used that $\theta \in [-\pi/2, \pi/2]$ and that $\cos\theta \geqslant 0$ for these values of θ.]

On the other hand. Let's compute the derivative of $\arcsin(x)$ directly from its definition as the inverse function of $\sin(x)$: take the derivative of both sides of $\sin(\arcsin(x)) = x$ to get $\cos(\arcsin(x))\arcsin'(x) = 1$. Hence

$$\arcsin'(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-\sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1-x^2}}.$$

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