Spotlight on the exponential function

Euler's constant, the natural base

Euler's constant $e = 2.71828182845904523\ldots$ is unavoidable in Calculus. For instance, starting with only division (which is all we need to define the function $1/x$), we obtain

$$\int \frac{1}{x}\mathrm{d}x = \log_e|x| + C.$$

Likewise, e^x is the only exponential whose derivative is itself. More professionally speaking, we have the following characterization of the exponential function:

(exponential function) e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x}2^x = \ln(2)\,2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.)
Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

The following is a **preview** of a series (infinite sum):

(preview of Taylor series) From the IVP above, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

This is the **Taylor series** for e^x at $x = 0$. More on these later!

Important note. We can indeed construct this infinite sum directly from $y' = y$ and $y(0) = 1$. To see this, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3}{3!} = \frac{x^2}{2!}$.

Example 78. Suppose we have capital 1 and that, annually, we are receiving $1 = 100\%$ interest. How much capital do we have at the end of a year, if $\frac{1}{n}$ interest is paid n times a year?

[For instance, $n = 12$ if we receive monthly interest payments.]

Solution. At the end of the year, we have $\left(1 + \frac{1}{n}\right)^n$.

For instance. Here are a few values spelled out:

$$\begin{aligned}
n = 1: &\quad \left(1 + \frac{1}{n}\right)^n = 2\\
n = 4: &\quad \left(1 + \frac{1}{n}\right)^n = 2.4414\ldots\\
n = 12: &\quad \left(1 + \frac{1}{n}\right)^n = 2.6130\ldots\\
n = 100: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7048\ldots\\
n = 365: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7145\ldots\\
n = 1000: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7169\ldots\\
n \to \infty: &\quad \left(1 + \frac{1}{n}\right)^n \to e = 2.71828\ldots
\end{aligned}$$

It is natural to wonder what happens if interest payments are made more and more frequently. As the entry for $n \to \infty$ shows, if we keep increasing n, then we will get closer and closer to $e = 2.7182818284590452\ldots$ in our bank account after one year.

Challenge. Can you evaluate the limit $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ using your Calculus I skills?

Euler's identity

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.
- The **real part** of $z = x + iy$ is x and we write $\mathrm{Re}(z) = x$.

  Likewise the **imaginary part** is $\mathrm{Im}(z) = y$.

  Observe that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 79. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

Comment. It follows that $\cos(x) = \mathrm{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \mathrm{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
In particular, we see from here that $\cos(x) = \cosh(ix)$ and $i\sin(x) = \sinh(ix)$ (or, equivalently, $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$).

Example 80. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

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Example 81. Which trig identity hides behind $e^{i(x+y)} = e^{ix} e^{iy}$?

Solution. We observe that

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i\sin(x+y) \\ e^{ix} e^{iy} &= [\cos(x) + i\sin(x)][\cos(y) + i\sin(y)] \\ &= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)). \end{aligned}$$

Comparing real and imaginary parts, we conclude that

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and
- $\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$.

Example 82. Which trig identity hides behind $e^{ix} e^{-ix} = 1$?

Solution. Note that

$$\begin{aligned} e^{ix} e^{-ix} &= [\cos(x) + i\sin(x)][\cos(-x) + i\sin(-x)] = [\cos(x) + i\sin(x)][\cos(x) - i\sin(x)] \\ &= \cos^2 x + \sin^2 x. \end{aligned}$$

Hence, $e^{ix} e^{-ix} = 1$ translates into Pythagoras' identity $\cos^2 x + \sin^2 x = 1$.