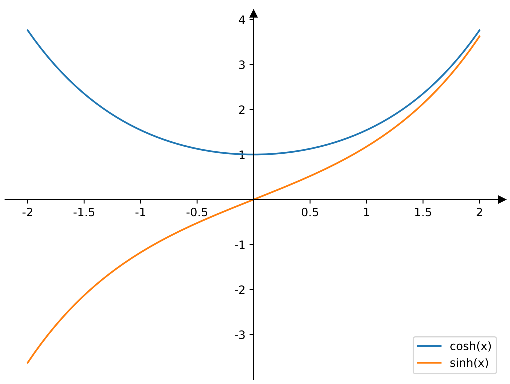
Hyperbolic functions

The **hyperbolic cosine and sine** are $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

The remaining hyperbolic trigonometric functions are built from these two as expected.

For instance, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.



We will later see that $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$. For now observe and verify the following properties that reflect similar properties of $\cos$ and $\sin$:

- $\cosh'(x) = \sinh(x)$
  $\sinh'(x) = \cosh(x)$
- $\cosh(-x) = \cosh(x)$ (that is, $\cosh$ is an even function)
  $\sinh(-x) = -\sinh(x)$ (that is, $\sinh$ is an odd function)
- $\cosh^2(x) - \sinh^2(x) = 1$

  This property explains the name **hyperbolic functions**: the points $(x, y) = (\cosh(t), \sinh(t))$ produce the **unit hyperbola** $x^2 - y^2 = 1$. This is analogous to how cosine and sine parametrize the circle: in that case, the points $(x, y) = (\cos(t), \sin(t))$ produce the **unit circle** $x^2 + y^2 = 1$.

  Comment. Circles and hyperbolas are conic sections (as are ellipses and parabolas).

  Comment. Plot the unit hyperbola. Then compare the graph to $y = \frac{1}{x}$. (This is a hyperbola, too!)

  Comment. Hyperbolic geometry plays an important role, for instance, in special relativity:
  https://en.wikipedia.org/wiki/Hyperbolic_geometry
- $e^x = \cosh(x) + \sinh(x)$

  This is a "cheap" version of **Euler's identity** $e^{ix} = \cos(x) + i\sin(x)$, which we will look at soon. In both cases, e^x and e^{ix} are broken up into their even part and odd part.

Example 74. Rewrite in terms of exponentials and simplify as much as possible:

(a) $4\sinh(\ln x)$

(b) $\cosh(3x) - \sinh(3x)$

Armin Straub
straub@southalabama.edu

Solution.

(a) $4\sinh(\ln x) = 4 \cdot \frac{e^{\ln x} - e^{-\ln x}}{2} = 2\left(x - \frac{1}{x}\right)$

(b) $\cosh(3x) - \sinh(3x) = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

Example 75. Determine the following:

(a) $\frac{\mathrm{d}}{\mathrm{d}x} 4\cosh(3x)$

(b) $\int 4\cosh(3x)\mathrm{d}x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(\sinh(x^2 + 3x))$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x} 4\cosh(3x) = 12\sinh(3x)$

(b) $\int 4\cosh(3x)\mathrm{d}x = \frac{4}{3}\sinh(3x) + C$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \ln(\sinh(x^2 + 3x)) = \frac{1}{\sinh(x^2 + 3x)} \cdot \cosh(x^2 + 3x) \cdot 2x$

Since the hyperbolic functions are defined in terms of the exponential function, it is not surprising that their inverse functions can be expressed in terms of logarithms. We leave it at the following example.

Example 76. Express $\sinh^{-1}$ in terms of logarithms.

Solution. We start with $y = \sinh(x) = \frac{e^x - e^{-x}}{2}$ and need to solve for x.
Write $u = e^x$ so that the equation becomes $2y = u - \frac{1}{u}$.
Multiplying with u and rearranging, we obtain $u^2 - 2yu - 1 = 0$ which is a quadratic equation in u.
Using the quadratic formula, we find $u = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}$. Note that $u = e^x > 0$ so that we have to choose the $+$ sign here.
Since $u = e^x$, this implies $x = \ln(u) = \ln\left(y + \sqrt{y^2 + 1}\right)$.
In summary, we have found that $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Comment. It follows from $\sinh(x) = -i\,\sin(ix)$ that $\arcsin = \sin^{-1}$ can be similarly expressed in terms of logarithms. However, we will now have the imaginary i in that formula.

Example 77. Find the length of the curve $y = \cosh x$ from $x = -2$ to $x = 2$.

Solution. The length is

$$\begin{aligned} \int_{-2}^{2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x &= \int_{-2}^{2} \sqrt{1 + \sinh^2(x)}\mathrm{d}x = \int_{-2}^{2} \cosh(x)\mathrm{d}x \\ &= \left[\sinh(x)\right]_{-2}^{2} = \sinh(2) - \sinh(-2) = 2\sinh(2) = e^2 - e^{-2} \approx 7.254. \end{aligned}$$

Here, we used that $1 + \sinh^2(x) = \cosh^2(x)$ as we had observed earlier (in the form $\cosh^2(x) - \sinh^2(x) = 1$).
Also note that $\cosh(x) > 0$ so that we get $\sqrt{\cosh^2(x)} = +\cosh(x)$.

Armin Straub
straub@southalabama.edu