**Example 65. (review)** Solve the initial value problem $\frac{dy}{dx} = xy$, $y(0) = 3$.

**Solution.** We separate variables to get $\frac{1}{y}dy = x\,dx$.

Integrating both sides, we find $\ln(y) = \frac{1}{2}x^2 + C$. (Since $y = 3$ in the initial condition, we don't need to write $\ln|y|$ because we have $y > 0$ around the initial condition.)

We can then find $C$ by using the values $x = 0$, $y = 3$ from the initial condition: $\ln(3) = \frac{1}{2} \cdot 0^2 + C$. So, $C = \ln(3)$.

We now solve $\ln(y) = \frac{1}{2}x^2 + \ln(3)$ for $y$ to get $y(x) = e^{\frac{1}{2}x^2 + \ln(3)} = 3e^{\frac{1}{2}x^2}$.

**Alternatively.** We could have also first solved for $y$ and then determined $C$ with the same result.

## Logarithms

**Review.** $\ln(x) = \log_e(x)$ is the inverse function of $e^x$. In other words, for all real $x$,

$$\ln(e^x) = x.$$

Similarly, $e^{\ln(x)} = x$ for all $x > 0$.

**Likewise.** $\log_a(x)$ is the inverse function of $a^x$ (where $a$ is called the base).

- $\frac{d}{dx}\ln(x) = \frac{1}{x}$

  **Why?** Start with $e^{\ln(x)} = x$ and differentiate both sides to get $e^{\ln(x)} \cdot \ln'(x) = 1$. It therefore follows that $\ln'(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$.

  **Note.** We also have $\frac{d}{dx}\ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$ for $x < 0$. Together, this implies the next entry.

- $\int \frac{1}{x}dx = \ln|x| + C$

The following observation allows us to convert other bases to the **natural base** $e$:

**(other bases)** $a^x = e^{\ln(a)x}$ and $\log_a(x) = \frac{\ln(x)}{\ln(a)}$.

**Why?** $a^x = e^{\ln(a)x}$ follows be writing $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$. Can you see how $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ follows from this?

Alternatively, the defining property of $\log_a$ is that $\log_a(a^x) = x$. Because $\ln(a^x) = x\ln(a)$ we see that $f(x) = \frac{\ln(x)}{\ln(a)}$ also has the property that $f(a^x) = x$.

**Example 66.** Compute $\frac{d}{dx}\log_a(x)$.

**Solution.** It follows from $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ that $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$.

**Example 67.** Compute $\frac{d}{dx}2^x$ and $\int 2^x dx$.

**Solution.** Write $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$ to see that $\frac{d}{dx}2^x = \ln(2)\,e^{x\ln(2)} = \ln(2)\,2^x$ and, likewise, $\int 2^x dx = \frac{1}{\ln(2)}e^{x\ln(2)} + C = \frac{2^x}{\ln(2)} + C$.

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**Example 68.** Rewrite $2^{\ln(x)}$ as a power of $x$.

**Solution.** We use the same "trick" as in the previous example (applying $e^{\ln(x)}$ to the expression) to get

$$2^{\ln(x)} = e^{\ln(2^{\ln(x)})} = e^{\ln(x)\ln(2)} = (e^{\ln(x)})^{\ln(2)} = x^{\ln(2)}.$$

**Alternatively.** Since $\ln(x) = \ln(2)\log_2(x)$, we have $2^{\ln(x)} = 2^{\ln(2)\log_2(x)} = (2^{\log_2(x)})^{\ln(2)} = x^{\ln(2)}$.

**Example 69.** Determine $\int \frac{2^{\ln(x)}}{x^2}\mathrm{d}x$.

**Solution.** By the previous example,

$$\int \frac{2^{\ln(x)}}{x^2}\mathrm{d}x = \int x^{\ln(2)-2}\mathrm{d}x = \frac{1}{\ln(2)-1}x^{\ln(2)-1} + C.$$

**Example 70.** Determine $\int \frac{(\ln x)^4}{3x}\mathrm{d}x$.

**Solution.** We substitute $u = \ln(x)$ in which case $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (so that the $\frac{1}{x}$ in the integrand will cancel out), to get

$$\int \frac{(\ln x)^4}{3x}\mathrm{d}x = \frac{1}{3}\int u^4 \mathrm{d}u = \frac{1}{15}u^5 + C = \frac{1}{15}(\ln x)^4 + C.$$

**Example 71.** Determine $\int \frac{\ln(\ln x)}{x \ln x}\mathrm{d}x$.

**Solution.** We substitute $u = \ln(x)$ in which case $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (so that the $\frac{1}{x}$ in the integrand will cancel out). Since

$$\int \frac{\ln(\ln x)}{x \ln x}\mathrm{d}x = \int \frac{\ln u}{u}\mathrm{d}u.$$

In the new integral, we substitute $v = \ln(u)$ with $\mathrm{d}v = \frac{1}{u}\mathrm{d}u$ to get

$$\int \frac{\ln(\ln x)}{x \ln x}\mathrm{d}x = \int \frac{\ln u}{u}\mathrm{d}u = \int v\mathrm{d}v = \frac{1}{2}v^2 + C = \frac{1}{2}(\ln u)^2 + C = \frac{1}{2}(\ln(\ln x))^2 + C.$$

**Example 72.** Determine $\int_0^2 3^{-x}\mathrm{d}x$.

**Solution.** Write $3^{-x} = e^{\ln(3^{-x})} = e^{-x\ln(3)}$ to see that

$$\int_0^2 3^{-x}\mathrm{d}x = \left[-\frac{1}{\ln(3)}e^{-x\ln(3)}\right]_0^2 = \left[-\frac{1}{\ln(3)}3^{-x}\right]_0^2 = \frac{1}{\ln(3)}(1-3^{-2}) = \frac{8}{9\ln(3)}.$$

**Example 73.** Determine $\int \frac{\log_3(x)}{x}\mathrm{d}x$.

**Solution.** Since $\log_3(x) = \frac{\ln(x)}{\ln(3)}$, we find that $\int \frac{\log_3(x)}{x}\mathrm{d}x = \frac{1}{\ln(3)}\int \frac{\ln(x)}{x}\mathrm{d}x$.

We now substitute $u = \ln(x)$ in which case $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ (so that the $\frac{1}{x}$ in the integrand will cancel out). Since

$$\int \frac{\ln(x)}{x}\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(x))^2 + C,$$

we get that $\int \frac{\log_3(x)}{x}\mathrm{d}x = \frac{1}{\ln(3)}\int \frac{\ln(x)}{x}\mathrm{d}x = \frac{1}{2\ln(3)}(\ln(x))^2 + B$ (where $B = \frac{C}{\ln(3)}$ is some constant).