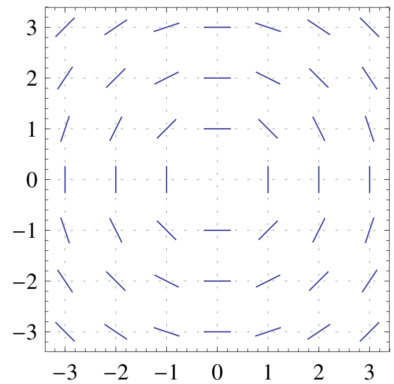
Slope fields, or sketching solutions to DEs

The next example illustrates that we can "plot" solutions to differential equations (it does not matter if we are able to actually solve them).

Comment. This is an important point because "plotting" really means that we can numerically approximate solutions. For complicated systems of differential equations, such as those used to model fluid flow, this is usually the best we can do. Nobody can actually solve these equations.

Example 57. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1, 2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1, 2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2} \frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Separation of variables, cont'd

Example 58. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{dy}{dx} = -\frac{x}{y}$.

Separate the variables to get $y\,dy = -x\,dx$ (in particular, we are multiplying both sides by dx).

Integrating both sides, we get $\int y\,dy = \int -x\,dx$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{dy}{dx}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Example 59. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found earlier in Example 58, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,dy = -x\,dx$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

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Example 60. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = 2$.

Solution. Proceeding as in the previous example, we find $y(x) = \sqrt{4 - x^2}$.

Example 61. Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$, $y(0) = 3$.

Solution. We separate variables to get $\frac{1}{y}\mathrm{d}y = x\mathrm{d}x$.

Integrating both sides, we find $\ln(y) = \frac{1}{2}x^2 + C$. (Since $y = 3$ in the initial condition, we don't need to write $\ln|y|$ because we have $y > 0$ around the initial condition.)
We can then find C by using the values $x = 0$, $y = 3$ from the initial condition: $\ln(3) = \frac{1}{2} \cdot 0^2 + C$. So, $C = \ln(3)$.
We now solve $\ln(y) = \frac{1}{2}x^2 + \ln(3)$ for y to get $y(x) = e^{\frac{1}{2}x^2 + \ln(3)} = 3e^{\frac{1}{2}x^2}$.
Alternatively. We could have also first solved for y and then determined C with the same result.

Which differential equations can we actually solve using separation of variables?

- A general DE of first order is typically of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$.

  For instance, $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy) - x^2 y$.

  Comment. First order means that only the first derivative of y shows up. The most general form of a DE of first order is $F(x, y, y') = 0$ but we can usually solve for y' to get to the above form.

- The ones we can solve are **separable equations**, which are of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$ (although simple) is not separable.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y - x}$ is separable because we can write it as $\frac{\mathrm{d}y}{\mathrm{d}x} = e^y e^{-x}$.

A very simple model of population growth

If $y(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{dy}{dt}$ might, from biological considerations, be (nearly) proportional to $y(t)$.

More down to earth, this is just saying "for a population 5 times as large, we expect 5 times as many babies".

Say, we have a population of $P = 100$ and $P' = 3$, meaning that the population changes by 3 individuals per unit of time. By how much do we expect a population of $P = 500$ to change? (Think about it for a moment!)

Without further information, we would probably expect the population of $P = 500$ to change by $5 \cdot 3 = 15$ individuals per unit of time, so that $P' = 15$ in that case. This is what it means for P' to be proportional to P. In formulas, it means that P'/P is constant or, equivalently, that $P' = kP$ for a proportionality constant k.

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ where k is the constant of proportionality.

The general solution to this DE is $y(t) = Ce^{kt}$. (Solve it yourself using separation of variables!)

Hence, mathematics tells us that populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 62. Let $y(t)$ describe the size of a population at time t. Suppose $y(0) = 100$ and $y(1) = 300$. Under the exponential model of population growth, find $y(t)$.

Solution. $y(t)$ solves the DE $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ and therefore is of the form $y(t) = Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k \cdot 0} = C = 100$ and $Ce^k = 100e^k = 300$. Hence $k = \ln(3)$ and $y(t) = 100e^{\ln(3)t} = 100 \cdot 3^t$.

Example 63. A yeast culture, with initial mass $12\ \mathrm{g}$, is assumed to exhibit exponential growth. After $10\ \mathrm{min}$, the mass is $15\ \mathrm{g}$. What is the mass after $t\ \mathrm{min}$?

Solution. Let $y(t)$ be the mass in g after $t\ \mathrm{min}$. $y(t)$ solves the DE $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ and so is of the form $y(t) = Ce^{kt}$.

We now use that $y(0) = 12$ and $y(10) = 15$ to determine both C and k.

$Ce^{k \cdot 0} = C = 12$ and $Ce^{10k} = 12e^{10k} = 15$. Hence $k = \frac{1}{10}\ln\left(\frac{5}{4}\right)$ and $y(t) = 12e^{\frac{1}{10}\ln\left(\frac{5}{4}\right)t} = 12\,(5/4)^{t/10}$.

Example 64. Just to give an indication of how the modelling can be refined, let us suppose we want to take limited resources into account, so that there is a maximum sustainable population size M. This situation could be modelled by the **logistic equation**

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right).$$

Note that if y is small (compared to M), then $1 - \frac{y}{M} \approx 1$, so $\frac{\mathrm{d}y}{\mathrm{d}t} \approx ky$, and we are back at our previous model. However, once the population is getting close to M then $1 - \frac{y}{M} \approx 0$, so $\frac{\mathrm{d}y}{\mathrm{d}t} \approx 0$, which means that the population does not continue to grow.

Main challenge of modeling: A model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

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