Differential equations

Example 50. The **differential equation** $\frac{dy}{dx} = y$ is solved by $y(x) = e^x$. It is also solved by $y(x) = 0$ and $y(x) = 7e^x$. Its general solution is $y(x) = Ce^x$ where C can be any number.

Example 51. The **initial value problem** $\frac{dy}{dx} = y$, $y(0) = 1$ has the unique solution $y(x) = e^x$.

The fact that the exponential function solves these simple equations is at the root of why it is so important!

Example 52. The general solution to the differential equation (DE) $\frac{dy}{dx} = x^2$ is $y(x) = \frac{1}{3}x^3 + C$.

In general, computing the antiderivative of $f(x)$ is the same as solving the (very special) DE $\frac{dy}{dx} = f(x)$.

Verifying if a function solves a DE

Given a function, we can always check whether it solves a DE!

We can just plug it into the DE and see if left and right side agree. This means that we can always check our work as well as that we can verify solutions generated by someone else (or a computer algebra system) even if we don't know the techniques for solving the DE.

Example 53. Consider the DE $\frac{dy}{dx} = y^2$.

(a) Is $y(x) = \frac{1}{x}$ a solution?

(b) $y(x) = -\frac{1}{x}$ a solution?

(c) Is $y(x) = -\frac{1}{x+3}$ a solution?

(d) Is $y(x) = 0$ a solution?

(e) Is $y(x) = 1$ a solution?

Solution.

(a) We compute $\frac{dy}{dx} = -\frac{1}{x^2}$. On the other hand, $y^2 = \frac{1}{x^2}$. Since $\frac{dy}{dx}$ and y^2 are not equal, $y = \frac{1}{x}$ is not a solution.

(b) We compute $\frac{dy}{dx} = \frac{1}{x^2}$. Since $y^2 = \left(-\frac{1}{x}\right)^2 = \frac{1}{x^2}$, we have $\frac{dy}{dx} = y^2$. Hence, $y = -\frac{1}{x}$ is a solution.

(c) We compute $\frac{dy}{dx} = \frac{1}{(x+3)^2}$. Since $y^2 = \frac{1}{(x+3)^2}$ as well, we conclude that $y = -\frac{1}{x+3}$ is another solution.

(d) Since $\frac{dy}{dx} = 0$ and $y^2 = 0$ as well, we again conclude that $y = 0$ is another solution.

(e) Since $\frac{dy}{dx} = 0$ while $y^2 = 1$, we conclude that $y = 1$ is not a solution.

Comment. We will solve this DE shortly and find that the general solution is $y(x) = -\frac{1}{x+C}$ (where the solution $y = 0$ corresponds to $C \to \infty$).

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Example 54. Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) We compute $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) We compute $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

Separation of variables

The next example demonstrates the method of **separation of variables** to solve (a certain class of) differential equations.

Example 55. Let us solve the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$ by separation of variables.

Comment. Some of the next steps might feel questionable... However, as illustrated above, we can always verify afterwards that we indeed found a solution.

In the first step, we separate the variables, including the differentials $\mathrm{d}y$ and $\mathrm{d}x$:

$$\frac{1}{y^2}\mathrm{d}y = \mathrm{d}x$$

[If the DE is of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$, then we would separate it as $\frac{1}{h(y)}\mathrm{d}y = g(x)\,\mathrm{d}x$.]

We then integrate both sides and compute the indefinite integrals:

$$\begin{aligned} \int \frac{1}{y^2}\mathrm{d}y &= \int \mathrm{d}x \\ -\frac{1}{y} &= x + C \qquad \text{[we combine the two constants of integration into one]} \end{aligned}$$

If possible (like here) we solve the resulting equation for y:

$$y = -\frac{1}{x + C}$$

Example 56. Find the general solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + y)e^x$, $y > -1$, using separation of variables.

Solution. We separate variables to get $\frac{1}{1+y}\mathrm{d}y = e^x\,\mathrm{d}x$.

Integrating both sides, we find $\ln(1 + y) = e^x + C$. (Since $y > -1$, we don't need to write $\ln|1 + y|$.)

We now solve for y to get $y(x) = e^{e^x + C} - 1$.

Exercise. As an exercise in differentiation, verify that $y(x)$ indeed solves the differential equation.

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