Review. Which equation describes the circle of radius r centered at the origin?

Solution. The circle consists of all points (x, y) that satisfy $x^2 + y^2 = r^2$.
This is just the Pythagorean theorem (make a sketch to make sure this is clear to you).

Example 37. We wish to compute the volume of a ball of radius r.

(a) Which region can we revolve to obtain a ball as our solid of revolution?

(b) Setup the appropriate integral for the volume and evaluate it.

For practice, you can compute it using disks/washers as well as using cylindrical shells.

Solution.

(a) We can revolve a half-circle to end up with a ball. A convenient choice is to take the region between $y = \sqrt{r^2 - x^2}$ and revolve it about the x-axis. This is what we will use for the next part.

(b) Taking vertical cross-sections, we get disks after revolving and the total volume is

$$\int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 \mathrm{d}x = \int_{-r}^{r} \pi(r^2 - x^2)\,\mathrm{d}x = \pi\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} = \frac{4}{3}\pi r^3$$

Sure enough, this is the formula for the volume of a ball that we have seen before (though our memory might be foggy on the exact formula).

Alternatively. It is more complicated here but, for practice, we can also take horizontal cross-sections in which case we get cylindrical shells after revolving. Lets take the cross-section at height y where y is between 0 and r. This cross-section extends from $x = -\sqrt{r^2 - y^2}$ to $x = \sqrt{r^2 - y^2}$ and therefore has width $2\sqrt{r^2 - y^2}$. Giving this slice a thickness of $\mathrm{d}y$ and revolving about the x-axis, we obtain a cylindrical shell with approximate volume

$$(\text{circumference}) \times (\text{width}) \times (\text{thickness}) = (2\pi y)\left(2\sqrt{r^2 - y^2}\right)\mathrm{d}y.$$

"Summing" all these volumes, we get

$$\int_0^r (2\pi y)\left(2\sqrt{r^2 - y^2}\right)\mathrm{d}y = 4\pi\int_0^r y\sqrt{r^2 - y^2}\,\mathrm{d}y = 4\pi\left[-\frac{1}{3}(r^2 - y^2)^{3/2}\right]_0^r = \frac{4}{3}\pi r^3.$$

Here, we substituted $u = r^2 - y^2$ to compute the integral (do it!). The final volume is, of course, the same we calculated before.

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Arc length

Example 38. What is the length of the curve $y = 2x$, for $0 \leqslant x \leqslant 4$?

Make a sketch and use Pythagoras.

Solution. The curve is the hypothenuse of a right triangle with shorter sides of length 4 (in x-direction) and 8 (in y-direction). Therefore its length is $\sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$.

(arc length) The length of a general curve $y = f(x)$, for $a \leqslant x \leqslant b$, is given by

$$\int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x.$$

Why? To see how we can arrive at this formula, we proceeded as follows:

- We chop the x-axis into little pieces of width $\mathrm{d}x$ and look at the corresponding pieces of our graph.
- Suppose we are looking at our graph near x. If we zoom in plenty, then the tiny portion of the graph we see begins to look roughly like a line with slope $f'(x)$.
- We can compute the length of a segment of this line as we did in Example 38 by using Pythagoras. If the segment extends $\mathrm{d}x$ horizontally, then it extends $f'(x)\mathrm{d}x$ vertically (make a sketch!). [We can also write $f'(x)\mathrm{d}x = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \mathrm{d}y$.]

  By Pythagoras, our piece of the line has length

  $$\sqrt{(\mathrm{d}x)^2 + (f'(x)\,\mathrm{d}x)^2} = \sqrt{1 + (f'(x))^2}\, \mathrm{d}x.$$

- "Adding" all these little pieces, we obtain the formula above for the total length of the curve.

Example 39. (again) Using the integral formula, compute the length of the curve $y = 2x$, for $0 \leqslant x \leqslant 4$, again. Of course, the answer agrees with Example 38.

Solution. Here, $f(x) = 2x$ so that $f'(x) = 2$. Hence, the length is

$$\int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x = \int_0^4 \sqrt{1 + 2^2}\, \mathrm{d}x = 4\sqrt{5} \approx 8.944.$$

Example 40. Compute the length of the curve $y = x^{3/2}$, for $0 \leqslant x \leqslant 4$.

Before you compute the answer, make a sketch. Which curve should be longer: $y = x^{3/2}$ or $y = 2x$?

Solution. With $f(x) = x^{3/2}$, we have $f'(x) = \frac{3}{2}\sqrt{x}$. The length is

$$\int_0^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, \mathrm{d}x = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, \mathrm{d}x = \int_1^{10} \frac{4}{9}\sqrt{u}\, \mathrm{d}u = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{8}{27}(\sqrt{1000} - 1) \approx 9.073.$$

Here, we substituted $u = 1 + \frac{9}{4}x$. Make sure that this substitution is clear to you (including the change of boundaries: if $x = 4$ then $u = ?$).

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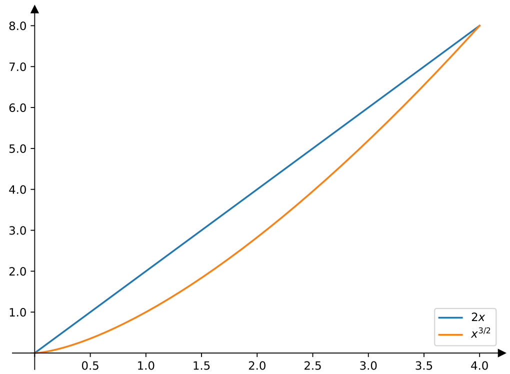
Comparing the lengths. As for comparing the lengths, note that, on the interval $0 \leqslant x \leqslant 4$, both $y = x^{3/2}$ and $y = 2x$ begin at the point $(0, 0)$ and end at the point $(4, 8)$. Since a line is the shortest connection between two points, the arc length for $y = x^{3/2}$ had to be larger.

However, you can see that the difference is not much. This is confirmed by the plot below:



Example 41. Setup an integral for the circumference of a circle of radius r.