Volumes using cylindrical shells

See Section 6.2 in our book for nice illustrations of cylindrical shells!

Example 34. Consider the region bounded by the curves $y = x^2$, $y = 0$ and $x = 2$. Determine the volume of the solid generated by revolving this region about the y-axis

(a) using horizontal cross-sections of the region (turning into washers), and

(b) using vertical cross-sections of the region (turning into **cylindrical shells**).

Solution. First, make a sketch as we did in class!

(a) Our region extends from $y = 0$ to $y = 4$. The cross-section at y (a line which extends from $x = \sqrt{y}$ to $x = 2$) turns into a washer with outer radius $R(y) = 2$ and inner radius $r(y) = \sqrt{y}$, which has area $\pi R(y)^2 - \pi r(y)^2 = \pi(4 - y)$. If we give this disk a thickness of $\mathrm{d}y$, then its volume is $\pi(4 - y)\mathrm{d}y$. The total volume is

$$\text{vol} = \int_0^4 \pi(4 - y)\,\mathrm{d}y = \pi\left[4y - \frac{1}{2}y^2\right]_0^4 = 8\pi.$$

(b) Our region extends from $x = 0$ to $x = 2$. The cross-section at x (a line which extends from $y = 0$ to $y = x^2$) turns into a cylindrical shell with radius $r(x) = x$ and height $h(x) = x^2$ and so has area $2\pi r(x) \cdot h(x) = 2\pi x^3$. If we give this shell a thickness of $\mathrm{d}x$, then its volume is $2\pi x^3 \mathrm{d}x$. The total volume is

$$\text{vol} = \int_0^2 2\pi x^3\,\mathrm{d}x = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi.$$

The following is a similar but slightly beefed up example.

Example 35. Consider the region bounded by the curves $y = x^2$, $y = 1$ and $x = 2$. Determine the volume of the solid generated by revolving this region about the line $x = -1$.

(a) using horizontal cross-sections of the region (turning into disks), and

(b) using vertical cross-sections of the region (turning into cylindrical shells).

Solution. As always, start with a sketch! Note that $y = 1$ is now describing the bottom: the bottom-left corner of the region is $(1, 1)$, the bottom-right corner is $(2, 1)$, and the top-right corner is $(2, 4)$.

(a) Our region extends from $y = 1$ to $y = 4$. The cross-section at y (a line which extends from $x = \sqrt{y}$ to $x = 2$) turns into a washer with outer radius $R(y) = 2 - (-1) = 3$ and inner radius $r(y) = \sqrt{y} - (-1) = \sqrt{y} + 1$, which has area $\pi R(y)^2 - \pi r(y)^2 = 9\pi - \pi(\sqrt{y} + 1)^2 = \pi(8 - 2\sqrt{y} - y)$. If we give this disk a thickness of $\mathrm{d}y$, then its volume is $\pi(8 - 2\sqrt{y} - y)\mathrm{d}y$. The total volume is

$$\text{vol} = \int_1^4 \pi(8 - 2\sqrt{y} - y)\,\mathrm{d}y = \pi\left[8y - \frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_1^4 = \pi\left(\frac{40}{3} - \frac{37}{6}\right) = \frac{43}{6}\pi.$$

(b) Our region extends from $x = 1$ to $x = 2$. The cross-section at x (a line which extends from $y = 1$ to $y = x^2$) turns into a cylindrical shell with radius $r(x) = x - (-1) = x + 1$ and height $h(x) = x^2 - 1$ and so has area $2\pi r(x) \cdot h(x) = 2\pi(x + 1)(x^2 - 1)$. If we give this shell a thickness of $\mathrm{d}x$, then its volume is $2\pi(x + 1)(x^2 - 1)\mathrm{d}x$. The total volume is

$$\text{vol} = \int_1^2 2\pi(x + 1)(x^2 - 1)\,\mathrm{d}x = \int_1^2 2\pi(x^3 + x^2 - x - 1)\,\mathrm{d}x = 2\pi\left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 - x\right]_1^2 = \frac{43}{6}\pi.$$

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Example 36. (extra) Consider the region enclosed by the curves

$$y = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 4.$$

Determine the volume of the solid generated by revolving this region about the x-axis

(a) using vertical cross-sections of the region, and

(b) using horizontal cross-sections of the region.

Solution. As always, start with a sketch!

(a) The volume is

$$\int_0^4 \pi(\sqrt{x})^2 \,\mathrm{d}x = \pi \int_0^4 x \,\mathrm{d}x = \pi\left[\frac{1}{2}x^2\right]_0^4 = 8\pi.$$

(b) We operate between $y = 0$ and $y = \sqrt{4} = 2$. A slice at height y has width $4 - y^2$ (note that $y = \sqrt{x}$ implies $x = y^2$). Giving this slice a thickness of $\mathrm{d}y$ and revolving about the x-axis, we obtain a cylindrical shell with approximate volume

$$(\text{circumference}) \times (\text{width}) \times (\text{thickness}) = (2\pi y)(4 - y^2) \,\mathrm{d}y.$$

"Summing" all these volumes, we get

$$\int_0^2 2\pi y(4 - y^2) \,\mathrm{d}y = 2\pi\left[2y^2 - \frac{1}{4}y^4\right]_0^2 = 2\pi(8 - 4) = 8\pi.$$

Sure enough, the volume is exactly what we calculated before.

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