Volumes using cross-sections

We have computed areas of certain regions by cutting them into tiny (typically vertical) pieces of width $\mathrm{d}x$ and some height $h(x)$. If the region extends from $x=a$ to $x=b$, then the total area is the sum of the areas of these pieces (each has area $h(x)\mathrm{d}x$) and therefore given by

$$\text{area} = \int_a^b h(x)\mathrm{d}x.$$

We now use the same idea to compute volumes:

Please have a look at the Section 6.1 in the book for all the pretty pictures and detailed explanations. Below is just a summary which probably doesn't make too much sense unless you have been in class or have read through the beginning of Section 6.1.

- The volume of a cylindrical solid is its base area times its height.
- The idea for computing the volume of more general solids is to cut it into little slices that are approximately cylindrical.
- Suppose that the slice at position x has cross-sectional area $A(x)$. If we are slicing with a width of $\mathrm{d}x$, then this slice has roughly volume $A(x)\mathrm{d}x$.
- Summing the volumes of all these slices leads to the follwing formula of the volume of the general solid:

$$\text{vol} = \int_a^b A(x)\mathrm{d}x$$

Note: It is usually up to us to introduce coordinates for the position x. The formula above assumes that our solid extends from $x=a$ to $x=b$ in these coordinates.

Example 30. Derive the formula for the volume of a pyramid of height h whose base is a square with sides of length a.

You might remember that the volume is $\frac{1}{3}a^2h$ but the point of this example is that we can actually find this formula without knowing it by slicing the pyramid (the easiest way to slice is horizontally, so that each cross-section is again just a square).

Solution. We cut the pyramid into cross-sections parallel to its base. Let x, between 0 and h, denote how far we have gone from the tip $(x=0)$ down to the base of the pyramid $(x=h)$. Then, at x, the cross-section is a square with side length $a\frac{x}{h}$ (make a sketch!). This cross-section has area $A(x) = \left(\frac{a}{h}x\right)^2$. Therefore, the volume is

$$\text{vol} = \int_0^h A(x)\mathrm{d}x = \int_0^h \left(\frac{a}{h}x\right)^2 \mathrm{d}x = \frac{a^2}{h^2}\int_0^h x^2\mathrm{d}x = \frac{a^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h = \frac{a^2}{h^2}\cdot\left(\frac{1}{3}h^3 - 0\right) = \frac{1}{3}a^2h.$$

Comment. Introducing the extra variable x can be done in different ways and it requires some practice to make the easiest choice. For instance, if we had let x, again between 0 and h, denote how far we have gone from the base $(x=0)$ to the tip of the pyramid $(x=h)$, then, at x, the cross-section is a square with side length $a\frac{x-h}{h}$. We would get the same in the end but with a little bit more algebra.

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Solids of revolution

Consider a region (for instance, the region enclosed by a bunch of curves). A solid of revolution is what we obtain when revolving this region about a given line.

Again, Section 6.1 in the book contains lots of helpful illustrations.

- Suppose the region is the area between a curve $R(x)$ and the x-axis, between $x=a$ and $x=b$.
- Further, suppose that we revolve this region about the x-axis. Then, slicing vertically, the cross-sections are disks (circles with the interior) with radius $R(x)$ (so that the cross-sectional area is $A(x)=\pi R(x)^2$).
- Hence, the volume of the resulting solid is

$$\mathrm{vol}=\int_a^b A(x)\mathrm{d}x=\int_a^b \pi R(x)^2\,\mathrm{d}x.$$

Example 31. Consider the region enclosed by the curves $y=\sqrt{x}$, $y=0$, $x=1$, $x=4$. If we revolve this region about the x-axis, what is the volume of the resulting solid?

Solution. Make a sketch! The solid should look roughly like a solid coffee mug (no room for coffee) without handles.

The cross-section at x is a disk with radius $R(x)=\sqrt{x}$ and so has area $\pi R(x)^2=\pi x$. If we give this disk a thickness of $\mathrm{d}x$, then its volume is $\pi R(x)^2\,\mathrm{d}x=\pi x\mathrm{d}x$. The total volume is

$$\mathrm{vol}=\int_1^4 \pi R(x)^2\,\mathrm{d}x=\int_1^4 \pi x\,\mathrm{d}x=\pi\left[\frac{1}{2}x^2\right]_1^4=\pi\left(8-\frac{1}{2}\right)=\frac{15}{2}\pi.$$

Example 32. Consider the region enclosed by the curves $y=\sqrt{x}$, $y=1$, $x=1$, $x=4$. If we revolve this region about the line $y=1$, what is the volume of the resulting solid?

Solution. Again, make a sketch! This solid should look roughly like a bullet.

The cross-section at x now is a disk with radius $R(x)=\sqrt{x}-1$ and so has area $\pi R(x)^2=\pi(\sqrt{x}-1)^2$. If we give this disk a thickness of $\mathrm{d}x$, then its volume is $\pi R(x)^2\,\mathrm{d}x=\pi x\mathrm{d}x$. The total volume is

$$\mathrm{vol}=\int_1^4 \pi R(x)^2\mathrm{d}x=\int_1^4 \pi\,(\sqrt{x}-1)^2\,\mathrm{d}x=\int_1^4 \pi\,(x-2\sqrt{x}+1)\,\mathrm{d}x=\pi\left[\frac{1}{2}x^2-\frac{4}{3}x^{3/2}+x\right]_1^4=\pi\left(\frac{4}{3}-\frac{1}{6}\right)=\frac{7}{6}\pi.$$

Example 33. Consider (again) the region enclosed by the curves $y=\sqrt{x}$, $y=1$, $x=1$, $x=4$. If we revolve this region about the x-axis, what is the volume of the resulting solid?

The solid should look roughly like the coffee mug from Example 31 with a cylindrical hole drilled out.

What do the cross-sections look like now?

Your final answer should be $\frac{9}{2}\pi$.

Solution. The solid should look roughly like the coffee mug from Example 31 with a cylindrical hole drilled out.

The cross-section at x now is a **washer** with outer radius $R(x)=\sqrt{x}$ and inner radius $r(x)=1$ so that its area is $\pi R(x)^2-\pi r(x)^2=\pi x-\pi=\pi(x-1)$. If we give this washer a thickness of $\mathrm{d}x$, then its volume is $\pi(x-1)\mathrm{d}x$. The total volume is

$$\mathrm{vol}=\int_1^4 \pi\,(x-1)\,\mathrm{d}x=\pi\left[\frac{1}{2}x^2-x\right]_1^4=\pi\left(4-\left(-\frac{1}{2}\right)\right)=\frac{9}{2}\pi.$$

Comment. In this simple case, we could also obtain the final answer from Example 31 by simply subtracting the volume of the cylinder (base area $\pi\cdot 1^2$ times height $4-1=3$ gives a volume of 3π) that is "drilled out".

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