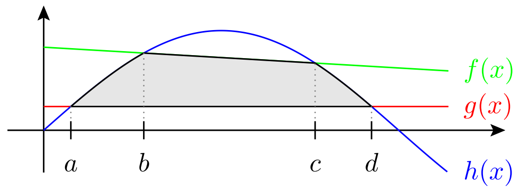
Example 27. Consider the plot below. What is the area enclosed by the curves $y = f(x)$, $y = g(x)$ and $y = h(x)$?



Solution. We split up the area into three smaller regions in which we can apply Theorem 25. The area is

$$\int_a^b [h(x) - g(x)]\mathrm{d}x + \int_b^c [f(x) - g(x)]\mathrm{d}x + \int_c^d [h(x) - g(x)]\mathrm{d}x.$$

Comment. In this case, we can alternatively (and equivalently) write the area as the difference

$$\int_a^d [h(x) - g(x)]\mathrm{d}x - \int_b^c [h(x) - f(x)]\mathrm{d}x.$$

Example 28. What is the area enclosed by the curves $y = 2 - x^2$, $y = -x$?

(a) First, make a sketch!

(b) Find intersections of the curves.

(c) Write down the integral for the area of interest.

(d) Evaluate the integral.

Solution.

(a) Do it! This should always be our first step.

(b) Since the equations for both curves are of the form $y = \ldots$, we can find the (x coordinates of the) intersections by setting the right-hand sides of the equations equal:

$$2 - x^2 = -x \quad \Longrightarrow \quad x^2 - x - 2 = 0 \quad \Longrightarrow \quad x = -1, 2$$

For the final step, we solved the quadratic equation (for instance, using the quadratic formula).

[It's not needed for the remaining parts, but we can get the corresponding y-coordinates from either $y = 2 - x^2$ or $y = -x$. The latter is simpler and we find that the two intersections are $(-1, 1)$ and $(2, -2)$.]

(c) This tells us (look at sketch!) that our area extends from $x = -1$ to $x = 2$ and that the curve $y = 2 - x^2$ is the upper boundary while $y = -x$ is the lower boundary. Therefore, the area is

$$\int_{-1}^2 ((2 - x^2) - (-x))\mathrm{d}x.$$

(d) We compute the area as

$$\int_{-1}^2 ((2 - x^2) - (-x))\mathrm{d}x = \int_{-1}^2 (2 + x - x^2)\mathrm{d}x = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^2 = \ldots = \frac{9}{2}.$$

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Example 29. What is the area enclosed by the curves $y=3-x^2$, $y=2$, $y=-1$?

First, make a sketch like we did in class!

For this problem, we then have a choice of whether we cut up our area into tiny vertical rectangles (with width $\mathrm{d}x$) or into tiny horizontal rectangles (with width $\mathrm{d}y$). Below, we do both. Of course, the final answer will be the same.

Solution. (vertical slicing) We first need to find the intersections of the parabola $y=3-x^2$ with each of $y=2$ and $y=-1$.

- $y=3-x^2$ and $y=2$: solving $3-x^2=2$, we get $x^2=1$ and so $x=\pm 1$.
- $y=3-x^2$ and $y=-1$: solving $3-x^2=-1$, we get $x^2=4$ and so $x=\pm 2$.

The area is therefore given by the following three integrals:

$$\int_{-2}^{-1}((3-x^2)-(-1))\mathrm{d}x+\int_{-1}^{1}(2-(-1))\mathrm{d}x+\int_{1}^{2}((3-x^2)-(-1))\mathrm{d}x$$

You might notice that the first and last integral must be equal, while the second integral is just computing the area of a rectangle of height $2-(-1)=3$ and width $1-(-1)=2$ (and so its area is $3\cdot 2=6$).

Whether or not you use any simplifications, the above integrals evaluate to $\frac{5}{3}+6+\frac{5}{3}=\frac{28}{3}$ (do it!).

Solution. (horizontal slicing) This time we don't need to compute the intersections because the area (see sketch!) clearly extends from $y=-1$ to $y=2$.

Since we are working in y-direction, we rewrite $y=3-x^2$ as $x^2=3-y$ or $x=\pm\sqrt{3-y}$ (note that $x=\sqrt{3-y}$ describes the right half, while $x=-\sqrt{3-y}$ describes the left half).

So the horizontal slice has length $\sqrt{3-y}-(-\sqrt{3-y})=2\sqrt{3-y}$. Accordingly, the total area is

$$\int_{-1}^{2}(\sqrt{3-y}-(-\sqrt{3-y}))\mathrm{d}y=2\int_{-1}^{2}\sqrt{3-y}\mathrm{d}y=2\left[-\frac{2}{3}(3-y)^{3/2}\right]_{-1}^{2}=-\frac{4}{3}(1-4^{3/2})=-\frac{4}{3}(1-8)=\frac{28}{3}.$$

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