Review. There will be a first quiz in lab on Thursday. One of the two problems will be to compute an integral like the following by substitution:

$$\int \cos^5(3t)\sin(3t)\mathrm{d}t$$

Example 24.

(a) Determine $\displaystyle\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t$.

(b) Determine $\displaystyle\int_0^\pi \frac{\sin^3(t)}{2-\cos(t)}\mathrm{d}t$.

Solution.

(a) **(again)** We substitute $u=2-\cos(t)$. When $t=0$ we have $u=2-\cos(0)=1$, and when $t=\pi$ we have $u=2-\cos(\pi)=3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{1}{u}\mathrm{d}u = \Big[\ln|u|\Big]_1^3 = \ln(3).$$

(b) We substitute $u=2-\cos(t)$ again. When $t=0$ we have $u=2-\cos(0)=1$, and when $t=\pi$ we have $u=2-\cos(\pi)=3$. This time, there will be $\sin^2(t)$ left over in the integral so we need to rewrite this in terms of u.

Note that $\cos(t)=2-u$. (At this point, we could solve for t to get $t=\arccos(2-u)$ and use this to substitute away any remaining t. However, we would get $\sin^2(t)=\sin^2(\arccos(2-u))$ which is not pleasant and would have to be simplified. We get this simplification for free by proceeding slightly differently.) Recall that $\cos^2(t)+\sin^2(t)=1$ so that $\sin^2(t)=1-\cos^2(t)=1-(2-u)^2=-u^2+4u-3$.

Therefore,

$$\int_0^\pi \frac{\sin^3(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{\sin^2(t)}{u}\mathrm{d}u = \int_1^3 \frac{-u^2+4u-3}{u}\mathrm{d}u = \int_1^3 \left(-u+4-\frac{3}{u}\right)\mathrm{d}u = \ldots = 4-3\ln(3).$$

Areas enclosed by curves

Theorem 25. The area enclosed by the curves $y=f(x)$ and $y=g(x)$, between $x=a$ and $x=b$, is given by

$$\int_a^b [f(x)-g(x)]\,\mathrm{d}x$$

provided that $f(x)\geqslant g(x)$ (for all $x\in[a,b]$).

[Note that the area is always $\int_a^b |f(x)-g(x)|\,\mathrm{d}x$ but to work with the absolute value, we need to break the problem into subcases according to whether $f(x)-g(x)\geqslant 0$ or $f(x)-g(x)\leqslant 0$.]

Example 26. What is the area enclosed by the curves $y=\cos(x)$, $y=1$, $x=0$, $x=2\pi$?
First, write down an integral and compute its value, then look at your sketch (always make a quick sketch!) and confirm that your answer makes rough sense.

Solution. Note that $y=1$ describes a horizontal line while $x=0$ and $x=2\pi$ are two vertical lines. For our area, $y=1$ lies above $y=\cos(x)$. Therefore, the area in question is

$$\int_0^{2\pi}(1-\cos(x))\mathrm{d}x = \Big[x-\sin(x)\Big]_0^{2\pi} = (2\pi-\sin(2\pi))-(0-\sin(0)) = 2\pi.$$

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