Substitution, cont'd

In general, finding the right substitution can be tricky and there are no straightforward rules that work for every integral. However, for our integrals there is usually a natural choice.

On the other hand, there is often more than one way to substitute successfully (see next example).

Example 14. (cont'd) Determine $\int x\sqrt{x^2+1}\mathrm{d}x$

(a) by substituting $u = x^2+1$, and

(b) by substituting $u = \sqrt{x^2+1}$.

(c) Explain why the substitution $u = x^2+1$ is particularly natural.

Solution.

(a) Since $u = x^2+1$, we have $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ so that $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
The integral therefore becomes

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

(b) Since $u = \sqrt{x^2+1}$, we have $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x}{u}$ so that $x\mathrm{d}x = u\mathrm{d}u$.
The integral therefore becomes

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int u\cdot u\,\mathrm{d}u = \int u^2\,\mathrm{d}u = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

(c) The $\sqrt{x^2+1}$ in the integrand will become $\sqrt{u}$ (of course) while the remaining x in the integrand is (up to a factor of 2) exactly the derivative of x^2+1 (and so will disappear when we bring in $\mathrm{d}u$ to replace $\mathrm{d}x$). With a bit of practice, this allows us to immediately see that our substitution will be a success.

Substitution in definite integrals

If we want to apply substitution in a definite integral like $\int_a^b f(x)\mathrm{d}x$, we have two options:

(a) We can first compute the indefinite integral $\int f(x)\mathrm{d}x$ using substitution.

If the result is $F(x) + C$, then $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$.

(b) Or, we can substitute the limits a and b as well to get a new definite integral $\int_c^d g(u)\mathrm{d}u$.

Often you can choose either approach. But for certain problems it might not be possible to compute the indefinite integral explicitly (so the first approach won't work), yet it can be useful obtain a substituted new integral using the second approach.

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Example 15. Determine $\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x$.

Solution. Let us illustrate both of the two approaches just mentioned:

(a) In the previous example, we already worked out that $\int x\sqrt{x^2+1}\,\mathrm{d}x = \frac{1}{3}(x^2+1)^{3/2} + C$. It follows that

$$\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x = \left[\frac{1}{3}(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}2^{3/2} - \frac{1}{3}.$$

(b) As in the previous example, we will substitute $u = x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we get $x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$. Moreover, if $x=0$ (lower limit) then $u = x^2+1 = 0^2+1 = 1$. And if $x=1$ (upper limit) then $u = x^2+1 = 1^2+1 = 2$. The definite integral therefore becomes

$$\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x \;=\; \int_1^2 \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \left[\frac{1}{3}u^{3/2}\right]_1^2 = \frac{1}{3}2^{3/2} - \frac{1}{3}.$$

Comment. While it looks like the first approach was much quicker, that's only because we had already computed the indefinite integral in the previous example. In general, if we haven't already done so, the second approach is a bit more direct (but requires us to pay attention to the limits of the integral).

Review 16. $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$

To verify, use $\ln|x| = \begin{cases} \ln(x), & \text{if } x>0, \\ \ln(-x), & \text{if } x<0, \end{cases}$ to differentiate the right-hand side.

Example 17. Determine $\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t$.

Here, a very natural substitution is $u = 2-\cos(t)$.

(Note how we can already anticipate that the derivative will nicely take care of the $\sin(t)$ in the integrand.)

Solution. We again have the choice of either substituting without limits or with limits:

(a) We substitute $u = 2-\cos(t)$. Since $\frac{\mathrm{d}u}{\mathrm{d}t} = \sin(t)$, we get $\sin(t)\mathrm{d}t = \mathrm{d}u$. Hence,

$$\int \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int \frac{1}{u}\mathrm{d}u = \ln|u| + C = \ln|2-\cos(t)| + C.$$

Hence (by the Fundamental Theorem of Calculus)

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \Big[\ln|2-\cos(t)|\Big]_0^\pi = \ln(3).$$

(b) Alternatively, once we feel comfortable with integration, we can do this substitution in a single step by also adjusting the boundaries. When $t=0$ we have $u = 2-\cos(0) = 1$, and when $t=\pi$ we have $u = 2-\cos(\pi) = 3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{1}{u}\mathrm{d}u = \Big[\ln|u|\Big]_1^3 = \ln(3).$$

Extra practice

Example 18. Determine $\int x^3\sqrt{x^2+1}\mathrm{d}x$.

Solution. We substitute $u=x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x}=2x$, we have $x\mathrm{d}x=\frac{1}{2}\mathrm{d}u$.
Note that there will be x^2 left into the integral which we replace with $x^2=u-1$.
The integral therefore becomes:

$$\begin{aligned}\int x^3\sqrt{x^2+1}\mathrm{d}x &= \int \frac{1}{2}x^2\sqrt{u}\,\mathrm{d}u=\int \frac{1}{2}(u-1)\sqrt{u}\,\mathrm{d}\\ &= \frac{1}{2}\int (u^{3/2}-u^{1/2})\,\mathrm{d}u=\frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}+C\\ &= \frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+C\end{aligned}$$

Comment. If you prefer, the final answer could be rewritten as $\frac{1}{15}(x^2+1)^{3/2}(3x^2-2)+C$.

Example 19. $\int \frac{\mathrm{d}x}{x\ln(x)}=$

First, use the substitution $u=\ln(x)$. (Can you already see why this is a good choice?)

Then, for practice, use $u=\frac{1}{\ln(x)}$ and see if you can get the same final answer.

(For more complicated integrals, finding the "best" substitution is quite an art form. For the integrals we are concerned with, there is always a natural choice.)

Example 20. $\int_2^4 \frac{\mathrm{d}x}{x\ln(x)}=$

To reduce work, use the previous problem and the fundamental theorem.

Example 21. $\int x\sin(x^2+3)\,\mathrm{d}x=$

Example 22. $\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)\mathrm{d}x=$

Example 23. $\int 3x^5\sqrt{x^3+1}\,\mathrm{d}x=$

First, use the substitution $u=x^3+1$ (that's the most natural choice).

Then, for practice, use $u=\sqrt{x^3+1}$ and see if you can get the same final answer.