Preparing for the Final

MATH 126 — Calculus II
Final Exam: Wednesday, Dec 11, 2024

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. A nongraphing calculator (equivalent to the TI-30XIIS) is allowed on the exam (but not needed). No notes or further tools of any kind will be permitted on the final exam.

Problem 1. Go over all past quizzes!

To help you with that, there is a version of each quiz posted on our course website without solutions (of course, there are solutions, too).

Problem 2. Study the practice problems for the three midterm exams!

Problem 3. Retake the three midterm exams!

(A copy without solutions is available on our course website. Of course, you also find solutions there.)

Additional problems covering the material since the third midterm

Problem 4.

(a) What are the Cartesian coordinates of the point with polar coordinates $r = 3$, $\theta = \frac{\pi}{6}$?

(b) What are the polar coordinates of the points with Cartesian coordinates $(-2, 2)$ and $(2, -2)$?

(c) Sketch the region described by $2 \leqslant r \leqslant 4$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{4}$.

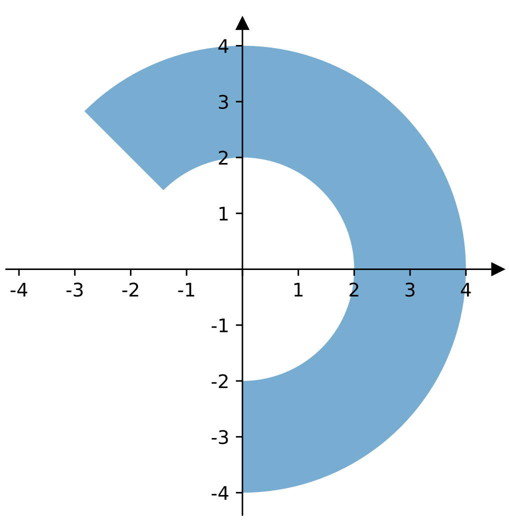
Solution.

(a) The Cartesian coordinates are $x = r\cos\theta = 3\cos\frac{\pi}{6} = 3\frac{\sqrt{3}}{2}$ and $y = r\sin\theta = 3\sin\frac{\pi}{6} = \frac{3}{2}$.

(b) Make a sketch! The polar coordinates of $(-2, 2)$ are $r = \sqrt{x^2 + y^2} = \sqrt{8} = 2\sqrt{2}$ and $\theta = \frac{3\pi}{4}$.

The polar coordinates of $(2, -2)$ are $r = \sqrt{x^2 + y^2} = \sqrt{8} = 2\sqrt{2}$ and $\theta = \frac{7\pi}{4}$ (or $\theta = -\frac{\pi}{4}$).

(c)



Armin Straub
straub@southalabama.edu

Problem 5. Consider the parametric curve given by $x = t\cos(2\pi t)$, $y = t\sin(2\pi t)$ with parameter $t \in [0, 3]$.

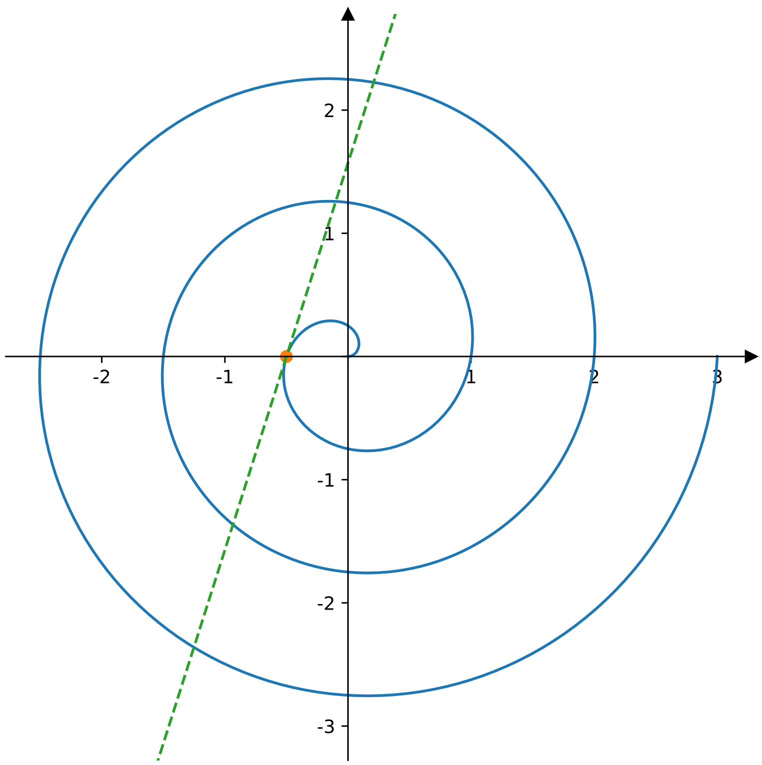
(a) Make a rough sketch of the curve.

(b) Find the slope of the line tangent to the curve at the point corresponding to $t = \frac{1}{2}$.

(c) Setup an integral for the arc length of the curve. Simplify but don't evaluate the integral.

Solution.

(a) This is a spiral starting at the origin (this is the point we get for $t = 0$) and completing exactly 3 turns to end at the point $(3, 0)$ on the x-axis (for $t = 3$). In the plot, we also highlight the tangent line of the next part.



(b) The tangent line has slope $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\sin(2\pi t) + t2\pi\cos(2\pi t)}{\cos(2\pi t) - t2\pi\sin(2\pi t)}$.

At our point, that is when $t = \frac{1}{2}$, the slope is $\frac{\sin(\pi) + \pi\cos(\pi)}{\cos(\pi) - \pi\sin(\pi)} = \frac{-\pi}{-1} = \pi$.

(c) The arc length of our curve is

$$\int_0^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^3 \sqrt{(\cos(2\pi t) - t2\pi\sin(2\pi t))^2 + (\sin(2\pi t) + t2\pi\cos(2\pi t))^2}\,\mathrm{d}t$$
$$= \int_0^3 \sqrt{1 + 4\pi^2 t^2}\,\mathrm{d}t.$$

Comment. The final integral could be evaluated using $\int \sqrt{1+t^2}\,\mathrm{d}t = \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\log\left(t + \sqrt{1+t^2}\right) + C$.

Problem 6.

(a) Write down the Taylor series for $7e^{-3x}$ at $x = 0$.

Armin Straub
straub@southalabama.edu

What is the radius of convergence of that series? What is the exact interval of convergence?

(b) Determine the Taylor series for $\int_0^x e^{-t^2/2}\,\mathrm{d}t$ at $x=0$.

What is the radius of convergence of that series? What is the exact interval of convergence?

Solution.

(a) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x (you need to know that!), we have

$$7e^{-3x} = 7\sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{7(-3)^n}{n!} x^n.$$

All these series converge for all x so that the radius of convergence is $R=\infty$. The interval of convergence is $(-\infty,\infty)$ (all real numbers).

Alternatively. If we didn't recall the Taylor series for e^x, we can work out the Taylor series of $f(x)=7e^{-3x}$ as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{7(-3)^n}{n!} x^n$$

(since $f^{(n)}(x)=7(-3)^n e^{-3x}$ and, hence, $f^{(n)}(0)=7(-3)^n$). We can then apply the ratio test with $a_n=\frac{7(-3)^n}{n!}x^n$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{7(-3)^{n+1}}{(n+1)!} x^{n+1} \cdot \frac{n!}{7(-3)^n x^n}\right| = 3|x|\frac{1}{n+1} \quad \longrightarrow \quad 0<1,$$

as $n\to\infty$, we conclude that the series converges for all x. Hence, $R=\infty$ and the interval of convergence is $(-\infty,\infty)$.

(b) Again, recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Hence, $e^{-t^2/2} = \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\,n!} t^{2n}$.

Integrating term by term, we find

$$\int_0^x e^{-t^2/2}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\,n!} \int_0^x t^{2n}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\,n!} \left[\frac{t^{2n+1}}{2n+1}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\,n!} \frac{x^{2n+1}}{2n+1}.$$

We could now apply the ratio test to this power series to find the radius of convergence (do it for extra practice!). However, when differentiating or integrating a power series, the radius of convergence R doesn't change. Hence, we still have $R=\infty$ (because that's what it is for e^x). The interval of convergence is $(-\infty,\infty)$.

Comment. Note how easy it was to compute the integral using power series. Note that our final power series is an antiderivative of $e^{-x^2/2}$. On the other hand, one can prove that this antiderivative cannot be expressed in terms of basic functions (exponentials, logarithms, trig functions, …). This means that, no matter how hard we try, it cannot be computed using the tools we have learned before power series.

Armin Straub
straub@southalabama.edu