Midterm #1

MATH 126 — Calculus II
Friday, Oct 2

Please print your name:

Problem 1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\mathrm{d}x}{x} =$

(b) $\displaystyle\int \sin(3x)\,\mathrm{d}x =$

(c) $\displaystyle\int \frac{\mathrm{d}x}{x^2+1} =$

Solution. $\displaystyle\int \frac{\mathrm{d}x}{x} = \ln|x| + C, \quad \int \sin(3x)\,\mathrm{d}x = -\frac{1}{3}\cos(3x) + C, \quad \int \frac{\mathrm{d}x}{x^2+1} = \arctan(x) + C$ □

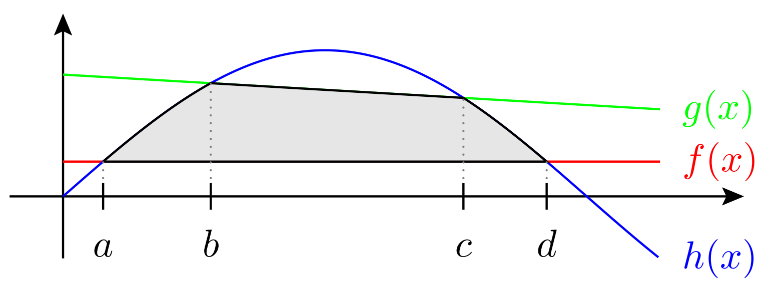
Problem 2. Determine the shape (but not the exact numbers involved) of the partial fraction decomposition of the following rational functions.

(a) $\displaystyle\frac{x}{x^2-1} =$

(b) $\displaystyle\frac{2x+1}{(x^2+1)^2(x+2)} =$

Solution. $\displaystyle\frac{x}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}, \quad \frac{2x+1}{(x^2+1)^2(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$ □

Problem 3. Consider the plot below. What is the area enclosed by the curves $y = f(x)$, $y = g(x)$ and $y = h(x)$? Your answer should be a sum of certain integrals.



Solution. The area is $\displaystyle\int_a^b [h(x) - f(x)]\mathrm{d}x + \int_b^c [g(x) - f(x)]\mathrm{d}x + \int_c^d [h(x) - f(x)]\mathrm{d}x.$ □

Problem 4. Set up an integral (but do not evaluate) for the length of the curve $y = x^3$ for $1 \leqslant x \leqslant 2$.

Solution.

$$\int_1^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_1^2 \sqrt{1 + (3x^2)^2}\,\mathrm{d}x = \int_1^2 \sqrt{1 + 9x^4}\,\mathrm{d}x$$

□

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Problem 5. Evaluate the integral $\int_1^2 x\ln(x)\,\mathrm{d}x$.

Solution. We apply integration by parts, and use $\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$ with $f(x) = \ln(x)$ and $g'(x) = x$. With $g(x) = \frac{1}{2}x^2$, we then get

$$\int_1^2 x\ln(x)\,\mathrm{d}x = \left[\frac{1}{2}x^2\ln(x)\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\frac{1}{2}x^2\,\mathrm{d}x = 2\ln(2) - \frac{1}{2}\int_1^2 x\,\mathrm{d}x = 2\ln(2) - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^2 = 2\ln(2) - \frac{3}{4}.$$

□

Problem 6. Evaluate the integral $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x$.

Solution. We substitute $u = x^3 + 1$, so that $\mathrm{d}u = 3x^2\mathrm{d}x$, and get

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\,\mathrm{d}u = \left[\frac{2}{3}u^{1/2}\right]_1^9 = 2 - \frac{2}{3} = \frac{4}{3}.$$

□

Problem 7. Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$, $y(0) = 1$.

Solution. We separate variables,

$$\frac{1}{y^2}\,\mathrm{d}y = \mathrm{d}x$$

and integrate

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int \mathrm{d}x$$

to find

$$-\frac{1}{y} = x + C.$$

Plugging in $y = 1$ and $x = 0$, we find $C = -1$. Solving for y, we find that

$$y(x) = -\frac{1}{x-1} = \frac{1}{1-x}.$$

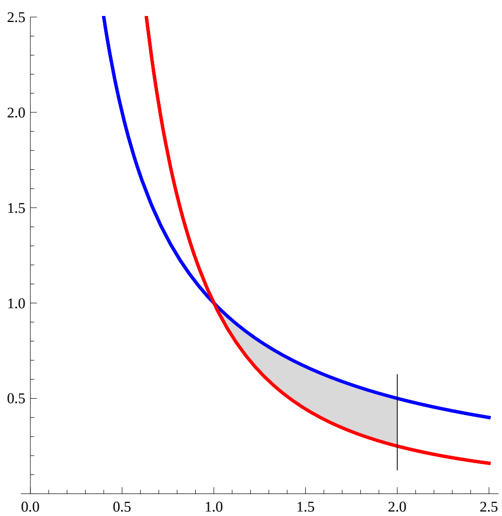
□

Problem 8. Set up an integral (but do not evaluate) for the volume of the solid obtained by revolving the region enclosed by the curves

$$y = \frac{1}{x}, \quad y = \frac{1}{x^2}, \quad x = 3,$$

about the line $y = -2$.

Solution. Here is a sketch:



The two curves intersect at $x = 1$ (this is clear from the sketch, or follows from equating $\frac{1}{x} = \frac{1}{x^2}$). The region therefore extends from $x = 1$ to $x = 3$. In that interval, $\frac{1}{x}$ is bigger than $\frac{1}{x^2}$. When revolving about the horizontal line $y = -2$, the volume is

$$\int_1^3 \left[\pi\left(\frac{1}{x} - (-2)\right)^2 - \pi\left(\frac{1}{x^2} - (-2)\right)^2\right]\mathrm{d}x.$$

Of course, it would be easy (if a bit annoying by hand) to evaluate this integral. □

Problem 9. (Bonus!) Roughly, what is the speed of light (in vacuum)?

Solution. Roughly, 300,000 km/s or 186,000 miles/s. □

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