Preparing for Midterm #1

MATH 126 — Calculus II
Friday, Oct 2

Please print your name:

Problem 1. Go over all the quizzes!

To help you with that, there is a version of each quiz posted on our course website without solutions (of course, there's solutions, too).

Problem 2. Find the length of the following curve:

$$y = 1 - 2x^{3/2}, \quad 0 \leqslant x \leqslant \frac{1}{3}.$$

Solution. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = -3\sqrt{x}$, the length of the curve is given by

$$\int_0^{1/3} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \,\mathrm{d}x = \int_0^{1/3} \sqrt{1+9x}\,\mathrm{d}x = \left[\frac{2}{3} \cdot \frac{1}{9}(1+9x)^{3/2}\right]_0^{1/3} = \frac{2}{27}(8-1) = \frac{14}{27}.$$

□

Problem 3. Evaluate $\int_0^2 \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x$.

Solution. We substitute $u = 4 - x^2$ so that $\mathrm{d}u = -2x\,\mathrm{d}x$ to get (if $x = 0$ then $u = 4$; if $x = 2$ then $u = 0$)

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int_4^0 \frac{\mathrm{d}u}{\sqrt{u}} = \left[-\frac{1}{2} \cdot 2\sqrt{u}\right]_4^0 = \sqrt{4} = 2.$$

□

Problem 4. Evaluate $\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}\,\mathrm{d}x$.

Solution. First, we do long division to obtain

$$\frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} = 1 - \frac{4}{x^3 - 2x^2}.$$

Next, we factor the denominator:

$$x^3 - 2x^2 = x^2(x-2).$$

We know that we can find numbers A, B, C such that

$$\frac{4}{x^3 - 2x^2} = \frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}.$$

Clearing denominators, we get

$$4 = x(x-2)A + (x-2)B + x^2 C.$$

Setting $x = 2$, we find $4 = 4C$, so $C = 1$. Setting $x = 0$, we find $4 = -2B$, so $B = -2$. There's no great third choice for x, so we plug in any value, say, $x = 1$ to find $4 = -A - B + C = -A + 2 + 1$. Hence, $A = -1$.

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Therefore,

$$\begin{aligned}\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,\mathrm{d}x &= \int_3^4 1+\frac{1}{x}+\frac{2}{x^2}-\frac{1}{x-2}\,\mathrm{d}x \\ &= \left[x+\ln|x|-\frac{2}{x}-\ln|x-2|\right]_3^4 \\ &= \frac{7}{6}+\ln\frac{2}{3}.\end{aligned}$$

□

Problem 5. Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2}{x^2+1}, \quad y(0)=2.$$

Solution. Using separation of variables we find

$$\int \frac{1}{y^2}\,\mathrm{d}y=\int \frac{1}{x^2+1}\,\mathrm{d}x$$

and therefore

$$-\frac{1}{y}=\arctan(x)+C.$$

Using the initial value $y=2$ when $x=0$, we get $-\frac{1}{2}=\arctan(0)+C=C$.

Finally, using $C=-\frac{1}{2}$ and solving for y, we find that the solution to the initial value problem is

$$y(x)=-\frac{1}{\arctan(x)-\frac{1}{2}}.$$

□

Problem 6. Evaluate $\displaystyle\int_{-2}^2 \frac{1}{x+1}\,\mathrm{d}x$ or show that the integral diverges.

Solution. The integrand has a singularity at $x=-1$. We therefore split

$$\int_{-2}^2 \frac{1}{x+1}\,\mathrm{d}x=\int_{-2}^{-1} \frac{1}{x+1}\,\mathrm{d}x+\int_{-1}^2 \frac{1}{x+1}\,\mathrm{d}x.$$

The first of these integrals is

$$\int_{-2}^{-1} \frac{1}{x+1}\,\mathrm{d}x=\lim_{b\to-1^-}\int_{-2}^{b} \frac{1}{x+1}\,\mathrm{d}x=\lim_{b\to-1^-}\ln|x+1|\Big|_{-2}^{b}=\lim_{b\to-1^-}\ln|b+1|=-\infty,$$

and thus diverges. This means that the integral $\displaystyle\int_{-2}^2 \frac{1}{x+1}\,\mathrm{d}x$ also diverges. □

Problem 7. Consider the region enclosed by the curves

$$y=\frac{1}{x}, \quad y=\frac{1}{x^2}, \quad x=2.$$

(a) Sketch the region and find its area.

(b) Find the volume of the solid obtained by revolving this region about the line $y=0$.

(c) Find the volume of the solid obtained by revolving this region about the line $y=-1$.

Solution.

(a) Here is a sketch:

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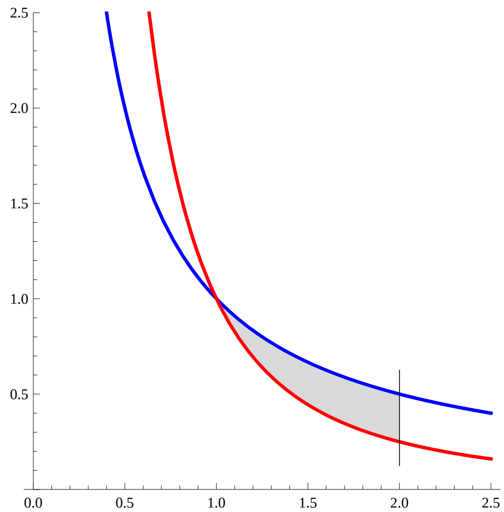

The two curves intersect at $x=1$ (this is clear from the sketch, or follows from equating $\frac{1}{x}=\frac{1}{x^2}$). The region therefore extends from $x=1$ to $x=2$. In that interval, $\frac{1}{x}$ is bigger than $\frac{1}{x^2}$. Therefore, the area is given by

$$\int_1^2 \left(\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x = \left[\ln|x|+\frac{1}{x}\right]_1^2 = \ln(2)+\frac{1}{2}-1 = \ln(2)-\frac{1}{2}.$$

(b) We are just revolving about the x-axis. The volume is

$$\int_1^2 \left[\pi\left(\frac{1}{x}\right)^2 - \pi\left(\frac{1}{x^2}\right)^2\right]\mathrm{d}x = \pi\int_1^2 \left(\frac{1}{x^2}-\frac{1}{x^4}\right)\mathrm{d}x = \pi\left[-\frac{1}{x}+\frac{1}{3}\frac{1}{x^3}\right]_1^2 = \pi\left(-\frac{11}{24}-\left(-\frac{2}{3}\right)\right) = \frac{5\pi}{24}.$$

(c) We are now revolving about the horizontal line $y=-1$. The volume is

$$\begin{aligned}
\int_1^2 \left[\pi\left(\frac{1}{x}-(-1)\right)^2 - \pi\left(\frac{1}{x^2}-(-1)\right)^2\right]\mathrm{d}x &= \pi\int_1^2 \left[\left(\frac{1}{x}+1\right)^2 - \left(\frac{1}{x^2}+1\right)^2\right]\mathrm{d}x \\
&= \pi\int_1^2 \left[\frac{1}{x^2}+\frac{2}{x}+1-\left(\frac{1}{x^4}+\frac{2}{x^2}+1\right)\right]\mathrm{d}x \\
&= \pi\int_1^2 \left[\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^4}\right]\mathrm{d}x \\
&= \pi\left[2\ln|x|+\frac{1}{x}+\frac{1}{3}\frac{1}{x^3}\right]_1^2 = \pi\left(2\ln(2)-\frac{19}{24}\right).
\end{aligned}$$

□

Problem 8. Evaluate $\int x^3\cos(x^2+1)\mathrm{d}x$.

Solution. First, we substitute $t=x^2+1$ (so that $\mathrm{d}t=2x\,\mathrm{d}x$) to get

$$\int x^3\cos(x^2+1)\mathrm{d}x = \frac{1}{2}\int x^2\cos(t)\mathrm{d}t = \frac{1}{2}\int (t-1)\cos(t)\mathrm{d}t.$$

Next, integrate by parts with $f(t)=t-1$ and $g'(t)=\cos(t)$. With $g(t)=\sin(t)$, we find

$$\int (t-1)\cos(t)\mathrm{d}t = (t-1)\sin(t) - \int \sin(t)\mathrm{d}t = (t-1)\sin(t)+\cos(t)+C.$$

Substituting back $t=x^2+1$, we finally obtain

$$\int x^3\cos(x^2+1)\mathrm{d}x = \frac{1}{2}(x^2\sin(x^2+1)+\cos(x^2+1))+C.$$

□

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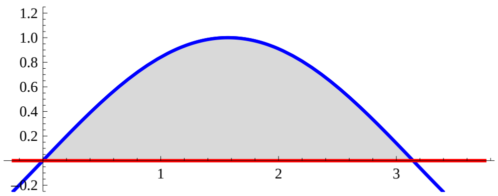
Problem 9. Consider the region bounded by the curves

$$y=0, \quad y=\sin(x), \quad 0\leqslant x\leqslant \pi.$$

Sketch the region, then set up an integral for the volume of the solid obtained by rotating this region about the x-axis. Evaluate this integral using integration by parts.

Solution.



The volume is

$$\int_0^\pi \pi \sin^2(x)\,\mathrm{d}x.$$

To evaluate this integral we integrate by parts with $f(x)=\sin(x)$ and $g'(x)=\sin(x)$. With $g(x)=-\cos(x)$ and $f'(x)=\cos(x)$, we find

$$\begin{aligned}\pi\int_0^\pi \sin^2(x)\,\mathrm{d}x &= \pi\left(-\cos(x)\sin(x)\Big|_0^\pi + \int_0^\pi \cos^2(x)\,\mathrm{d}x\right)\\ &= 0+\pi\int_0^\pi (1-\sin^2(x))\,\mathrm{d}x\\ &= \pi^2-\pi\int_0^\pi \sin^2(x)\,\mathrm{d}x.\end{aligned}$$

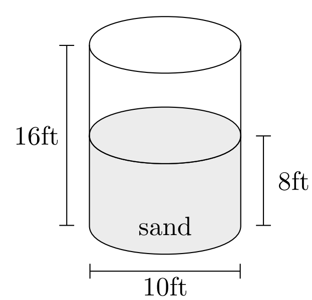
It follows that $2\pi\int_0^\pi \sin^2(x)\,\mathrm{d}x=\pi^2$, and hence the volume is $\pi\int_0^\pi \sin^2(x)\,\mathrm{d}x=\frac{\pi^2}{2}$. □

Problem 10. Consider the cylindrical container displayed to the right. It is half filled with sand weighing 100 lb/ft^3.

(a) Determine the amount of work needed to lift the sand to the rim of the tank.

(b) Determine the amount of work needed to lift the sand to a level 10 ft above the rim of the tank. Just an integral is good enough, here.

(c) Now, suppose the container is completely filled with sand. Determine the amount of work needed to lift the sand to a level 10 ft above the rim of the tank. Again, an integral is good enough, here.



Solution.

(a) Let x measure height (in ft) starting from the bottom of the container.

We consider a (horizontal) "slice" of the container at position x (and thickness $\mathrm{d}x$).

- The volume of this slice is $\mathrm{vol}=\pi\cdot 5^2\cdot \mathrm{d}x$ (ft^3). Its weight is $2500\pi\,\mathrm{d}x$ (lb).
- This slice needs to be lifted $16-x$ (ft).
- Thus, the work for this slice is $2500\pi(16-x)\,\mathrm{d}x$ (ft-lb).

There is slices from $x = 0$ to $x = 8$. “Adding” these up, we find that the total amount of work is

$$\text{work} = \int_0^8 2500\pi(16 - x)\,\mathrm{d}x = 2500\pi\left[16x - \frac{1}{2}x^2\right]_0^8 = 2500\pi \cdot 96 \approx 754,000\,\text{ft-lb}.$$

(b) The only adjustment to the first part is that each slice needs to be lifted $16 + 10 - x$ (ft) now. Hence, we find that the total amount of work is

$$\text{work} = \int_0^8 2500\pi(26 - x)\,\mathrm{d}x = 2500\pi \cdot 176 \approx 1,382,000\,\text{ft-lb}.$$

(c) The only further adjustment is that there are now slices from $x = 0$ to $x = 16$. The total amount of work is

$$\text{work} = \int_0^{16} 2500\pi(26 - x)\,\mathrm{d}x = 2500\pi \cdot 288 \approx 2,262,000\,\text{ft-lb}.$$

□