Review 193. Taylor series, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

[Note that the series for e^x and $\cos x$ converge for all x (radius of convergence ∞).]

Example 194. Determine the Taylor series of $f(x) = e^{2x}$ at $x = 0$.

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Solution. Observe that $f^{(n)}(x) = 2^n e^{2x}$. Hence, $e^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

Note. $e^{2x} = e^x e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots$

[For instance, we get the $\frac{4}{3}x^3$ as $1 \cdot \frac{x^3}{6} + x \cdot \frac{x^2}{2} + \frac{x^2}{2} \cdot x + \frac{x^3}{6} \cdot 1 = \frac{4}{3}x^3$.]

(Which matches the first terms of our series for e^{2x}.) This illustrates that we can multiply Taylor series.

Example 195. Determine the Taylor series of $\int e^{-x^2} \mathrm{d}x$ at $x = 0$.

Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$.

Integrating term by term, we conclude that $\int e^{-x^2} \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C$.

Note. Since e^{-x^2} is an even function, its Taylor series only includes the terms x^{2n} (which are even) and not terms of the form x^{2n+1} (which are odd). See also the Taylor series that we got for $\cos(x)$ (which is even).

Example 196. Find the first four terms of the Taylor series of $e^x \cos(x)$ at $x = 0$.

[Multiply the Taylor series for e^x and $\cos(x)$ as we did at the end of Example 194.]

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