Example 178. (yesterday) Determine convergence of $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\, 4^n}$ and $\sum_{n=1}^{\infty} \frac{5^n}{n^2\, 4^n}$.

Solution. Both series $\sum a_n$ diverge because $\lim_{n\to\infty} a_n \neq 0$.

Note. In the first case, limit comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ works to show that the series diverges.

However, this comparison is very "wasteful" and doesn't concentrate on the dominating terms. This is illustrated by the fact that limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (which converges) does not yield any conclusions for the second series.

Example 179. Consider the power series $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$

(a) Determine the radius of convergence R.

(b) Let $f(x) = \sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$ for x such that $|x-2| < R$. Write down series for $f'(x)$, $f''(x)$ and the indefinite integral $\int f(x) \mathrm{d}x$.

Solution.

(a) We apply the ratio test with $a_n = \frac{3^n (x-2)^n}{n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n (x-2)^n}\right| = 3|x-2| \frac{n}{n+1} \to 3|x-2| \text{ as } n \to \infty$$

The ratio test implies that $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$ converges if $3|x-2| < 1$ or, equivalently, $|x-2| < \frac{1}{3}$.

The radius of convergence therefore is $\frac{1}{3}$.

(b) $f'(x) = \sum_{n=1}^{\infty} \frac{3^n}{n} n (x-2)^{n-1} = \sum_{n=1}^{\infty} 3^n (x-2)^{n-1}$

$$f''(x) = \sum_{n=1}^{\infty} 3^n (n-1)(x-2)^{n-2} = \sum_{n=2}^{\infty} 3^n (n-1)(x-2)^{n-2}$$

$$\int f(x) \mathrm{d}x = \sum_{n=1}^{\infty} \frac{3^n}{n} \frac{(x-2)^{n+1}}{n+1} + C$$

Example 180. Evaluate $\sum_{n=1}^{\infty} 3^n (x-2)^{n-1}$ (from the previous problem) if $|x-2| < \frac{1}{3}$.

Solution. This is a geometric series. Here are two ways to rewrite it so we can apply $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

- $\sum_{n=1}^{\infty} 3^n (x-2)^{n-1} = 3 \sum_{n=1}^{\infty} 3^{n-1} (x-2)^{n-1} = 3 \sum_{n=0}^{\infty} 3^n (x-2)^n = 3 \cdot \frac{1}{1-3(x-2)} = \frac{3}{7-3x}$
- $\sum_{n=1}^{\infty} 3^n (x-2)^{n-1} = \frac{1}{x-2} \sum_{n=1}^{\infty} 3^n (x-2)^n = \frac{1}{x-2}\left[\frac{1}{1-3(x-2)} - 1\right] = \frac{3}{7-3x}$

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