Review. power series, radius of convergence, term-by-term differentiation and integration

Review. Determine the derivative of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. We have previously determined that this power series about $x=0$ has convergence radius $R=\infty$. Therefore, the function $f(x)$ is defined by the series for all $x \in \mathbb{R}$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

Important note. Observe that $f'(x) = f(x)$ and $f(0) = 1$. In other words, $f(x)$ is the unique solution of the IVP $y' = y$, $y(0) = 1$. We conclude that $f(x) = e^x$.

Note. Take the derivative of the first few terms of $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ to observe how the series reproduces itself when differentiated.

Note. If one continues this line of thinking, then one actually finds a powerful method for solving differential equations by finding power series solutions.

Example 175. Differentiate both sides of $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Solution. We find $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$. This identity is valid if $|x| < 1$ (why?).

Comment. If we prefer, we can also write $\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$.

Note. The new series $\sum_{n=1}^{\infty} n x^{n-1}$ has again radius of convergence 1 (like the geometric series).

This is a general phenomenon. Differentiating and integrating power series does not change the radius of convergence.

[You can see this by thinking about the effect of an additional factor of n in a_n when applying the ratio test.]

Example 176. Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution. The first series is just a geometric series: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1 = 1$

[also recall our geometric argument from when we started to learn about series]

For the second series, we use $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ with $x = \frac{1}{2}$. In that case, $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$.

Multiplying both sides by $\frac{1}{2}$, we obtain $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

Example 177. Find a power series (about $x=0$) for $\frac{1}{1+x^2}$.

[Our goal for next class is to integrate this power series to get our hands on $\arctan(x)$.]

Solution. We plug $-x^2$ for x in the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ to get

$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$. This is valid for $|-x^2| < 1$ or, equivalently, $|x| < 1$.

In particular, this power series has radius of convergence 1.

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