Example 172. What is the radius of convergence of the following power series?

(a) $\sum_{n=0}^{\infty} n^2 4^n (x+1)^n$ This is a power series about $x=-1$.

Solution. We apply the ratio test with $a_n = n^2 4^n (x+1)^n$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 4^{n+1}(x+1)^{n+1}}{n^2 4^n (x+1)^n}\right| = |4(x+1)| \frac{(n+1)^2}{n^2} \to |4(x+1)|$ as $n \to \infty$

The ratio test implies that $\sum_{n=0}^{\infty} n^2 4^n (x+1)^n$ converges if $|4(x+1)| < 1$ or, equivalently, $|x+1| < \frac{1}{4}$.

Radius of convergence is $\frac{1}{4}$.

Note. $\sum_{n=0}^{\infty} n^2 4^n (x+1)^n = \sum_{n=1}^{\infty} n^2 4^n (x+1)^n$ and it is only a matter of preference which of the two to use.

Theorem 173. (Term-by-term differentiation and integration)

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{on the interval } (a-R, a+R).$$

In this interval, $f(x)$ is arbitrarily often differentiable, and its derivatives can be obtained by differentiating the power series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

and so on. Likewise, $f(x)$ can be integrated term by term:

$$\int f(x) \mathrm{d}x = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

Example 174. Determine the derivative of $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. We have previously determined that this power series about $x=0$ has convergence radius $R=\infty$. Therefore, the function $f(x)$ is defined by the series for all $x \in \mathbb{R}$.

$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

Important note. Observe that $f'(x) = f(x)$ and $f(0) = 1$. In other words, $f(x)$ is the unique solution of the IVP $y' = y$, $y(0) = 1$. We conclude that $f(x) = e^x$.

Armin Straub
straub@southalabama.edu