Review. geometric series, ratio test

Let us redo Example 153 using the ratio test.

Example 163. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Solution. In this case $a_n = \frac{1}{n!}$, and so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} \to 0$ as $n \to \infty$.

Since $0 < 1$, the ratio test allows us to conclude that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

We now include an additional term in this series.

Example 164. Show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

[This is a generalization of Example 153 which considered the case $x = 1$. We will see later that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.]

Comment. Note that $\frac{x^n}{n!} \leqslant x^n$, so that (by direct comparison) our series converges for all x with $|x| < 1$. As we will see from the ratio test, our series actually converges for many more x (all of them!).

Solution. In this case $a_n = \frac{x^n}{n!}$, and so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0$ as $n \to \infty$.

Since $0 < 1$, the ratio test allows us to conclude that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

Important comment. We will see later that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$. This is the reason why we are so interested in understanding series. They allow us to represent functions that we care about in a new and useful way.

Comment. As a consequence, we see that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (for any x). Can you also give a direct argument?

Scary?

$$x^2 = \underbrace{x + x + \ldots + x}_{x \text{ times}} \quad \rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}x} x^2 = \frac{\mathrm{d}}{\mathrm{d}x}(x + x + \ldots + x) \quad \rightsquigarrow \quad 2x = 1 + 1 + \ldots + 1 = x \quad \rightsquigarrow \quad 2 = 1$$

[If you are bothered by the "x times", then note that the above can be written as $x^2 = xy$ with $y = x$. Differentiating both sides, we then have $2x = y$ or $2x = x$, and so $2 = 1$.
Can you see where we messed up?]

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