Example 157. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + \log(n)}$ converges.

Solution. Note that, for all $n \geqslant 1$, $n^2 + \log(n) \geqslant n^2$ and so $\frac{1}{n^2 + \log(n)} \leqslant \frac{1}{n^2}$.

By comparison, $\sum_{n=1}^{\infty} \frac{1}{n^2 + \log(n)} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} = \text{finite}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2 + \log(n)}$ converges.

[Note that you don't even want to spend time thinking about the corresponding integral $\int_1^{\infty} \frac{\mathrm{d}x}{x^2 + \log(x)}$. Its antiderivative cannot be written in terms of the functions we are familiar with.]

[Observe that we can just "see" this: for large n, our terms $\frac{1}{n^2 + \log(n)}$ "behave" like $\frac{1}{n^2}$ and so $\sum_{n=1}^{\infty} \frac{1}{n^2 + \log(n)}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. This reasoning is made precise by the limit comparison test that we learn about below. Redo this example using the limit comparison test!]

Example 158. Determine whether the series $\sum_{n=1}^{\infty} \frac{n + \log(n)}{n^2 - n}$ converges.

Solution. Note that, for all $n \geqslant 1$, $\frac{n + \log(n)}{n^2 - n} \geqslant \frac{n}{n^2 - n} > \frac{n}{n^2} = \frac{1}{n}$.

Hence, $\sum_{n=1}^{\infty} \frac{n + \log(n)}{n^2 - n} > \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, and so our series diverges.

[Observe that we can just "see" this: for large n, our terms $\frac{n + \log(n)}{n^2 - n}$ "behave" like $\frac{n}{n^2} = \frac{1}{n}$ and so $\sum_{n=1}^{\infty} \frac{n + \log(n)}{n^2 - n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n}$ converges. The next example makes this precise.]

Limit comparison test

Suppose that $a_n > 0$ and $b_n > 0$.

- If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then $\sum_{n=N}^{\infty} a_n$ and $\sum_{n=N}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=N}^{\infty} b_n$ converges, then $\sum_{n=N}^{\infty} a_n$ converges.
- If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=N}^{\infty} b_n$ diverges, then $\sum_{n=N}^{\infty} a_n$ diverges.

Observe that this makes a lot of sense: $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ with $c \neq 0$ means that the sequences $\{a_n\}$, $\{b_n\}$ grow in the same fashion (up to the overall factor c). Hence, their sums behave in the same fashion.

Example 159. Again, determine whether the series $\sum_{n=1}^{\infty} \frac{n + \log(n)}{n^2 - n}$ converges.

Solution. Let $a_n = \frac{n + \log(n)}{n^2 - n}$ and $b_n = \frac{1}{n}$ (this choice comes from the dominating terms in a_n; make sure you see this!). Since $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$, it follows that $\sum_{n=1}^{\infty} \frac{n + \log(n)}{n^2 - n}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

Since the latter diverges, we conclude that $\sum_{n=1}^{\infty} \frac{n + \log(n)}{n^2 - n}$ diverges.

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