Review 150. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Example 151. Determine whether the series $\sum_{n=1}^{\infty} \frac{\log n}{n}$ converges or diverges.

Solution. (via integral comparison) The integral $\int_1^{\infty} \frac{\log x}{x} \mathrm{d}x = \int_0^{\infty} u \mathrm{d}u$ obviously diverges (note that we substituted $u = \log x$), and hence $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges as well.

Comparison tests for series and integrals

Suppose that $a_n \geqslant 0$ and $b_n \geqslant 0$.

- If $a_n \leqslant b_n$ and $\sum_{n=N}^{\infty} b_n$ converges, then $\sum_{n=N}^{\infty} a_n$ converges.
- If $a_n \geqslant b_n$ and $\sum_{n=N}^{\infty} b_n$ diverges, then $\sum_{n=N}^{\infty} a_n$ diverges.

Example 152. Show, again, (this time by direct comparison) that the series $\sum_{n=1}^{\infty} \frac{\log n}{n}$ diverges.

Solution. Note that $\frac{\log n}{n} > \frac{1}{n}$ for all $n > 3$ (because $\log n > 1$ for $n > 3$).

But already $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty} \frac{\log n}{n}$ has to diverge as well.

[Note that $\sum_{n=3}^{\infty} \frac{\log n}{n} > \sum_{n=3}^{\infty} \frac{1}{n} = \infty$, and so our series diverges.]

Example 153. Show that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. This is the **factorial**.
[It counts the number of ways in which you can order n objects.]

Solution. Note that $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \geqslant n^2$ for all $n = 4, 5, 6, \ldots$
[In any case, $n!$ grows much faster than n^2, so that $n! \geqslant n^2$ and hence $\frac{1}{n!} \leqslant \frac{1}{n^2}$ for large enough n.]
Hence,

$$\sum_{n=4}^{\infty} \frac{1}{n!} \leqslant \sum_{n=4}^{\infty} \frac{1}{n^2},$$

and we already know that the right-hand side converges. Therefore, $\sum_{n=4}^{\infty} \frac{1}{n!}$ and hence $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, too.