Series

Remark 130. A tortoise racing a Greek hero... **Zeno's paradox**

https://en.wikipedia.org/wiki/Zeno%27s_paradoxes#Achilles_and_the_tortoise

Example 131. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$

Solution. Visual!

Solution. Redo this example, once you learn about the geometric series (below).

Example 132. In this example, we evaluate the sum $1+x+x^2+\ldots+x^M$.

- First, we note that $x(1+x+x^2+\ldots+x^M)=x+x^2+\ldots+x^M+x^{M+1}$, and that the result has most terms in common with our original sum.
- Hence, we look at the difference of $1+x+x^2+\ldots+x^M$ and $x(1+x+x^2+\ldots+x^M)$ to get

$$(1-x)(1+x+x^2+\ldots+x^M)=(1+x+x^2+\ldots+x^M)-(x+x^2+\ldots+x^M+x^{M+1})=1-x^{M+1}.$$

- Dividing both sides by $1-x$, we arrive at the **geometric sum**

$$\sum_{n=0}^{M}x^n=1+x+x^2+\ldots+x^M=\frac{1-x^{M+1}}{1-x}.$$

Taking the limit $M\to\infty$ in the geometric sum, we get: (recall that $\lim_{M\to\infty}x^M=0$ if $|x|<1$)

If $|x|<1$, then $\sum_{n=0}^{\infty}x^n=1+x+x^2+\ldots=\frac{1}{1-x}$. **(Geometric series)**

Example 133. Compute the following series:

(a) $\sum_{n=0}^{\infty}\frac{1}{2^n}=$

The final answer should be 2.

(b) $\sum_{n=1}^{\infty}\frac{1}{2^n}=$

Note that $\sum_{n=1}^{\infty}\frac{1}{2^n}=\sum_{n=0}^{\infty}\frac{1}{2^n}-\frac{1}{2^0}$. The final answer should be 1 (which is what our visual proof produced).

(c) $\sum_{n=0}^{\infty}\frac{7}{10^n}=$

The final answer should be $\frac{70}{9}$.

(d) $\sum_{n=2}^{\infty}\frac{7}{10^n}=$

The final answer should be $\frac{7}{90}$.

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