Example 115. The following example illustrates that limits of the form "$\frac{\infty}{\infty}$" are completely undetermined. Anything is possible for the actual limit:

- $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$
- $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to\infty} \frac{x}{3x} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}$
- $\lim_{x\to\infty} \frac{x(1+\sin^2(x))}{x} = \lim_{x\to\infty} (1+\sin^2(x))$ This limit does not exist.

Theorem 116. (L'Hospital's rule) If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

The same conclusion holds if $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

[It is important to realize that L'Hospital's rule only applies to the undetermined cases "$\frac{\infty}{\infty}$" and "$\frac{0}{0}$".]

Example 117. $\int_0^\infty x e^{-x} \mathrm{d}x =$

Your final answer should be 1.

Along the way, you will need the limit $\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$.

Sequences

A **sequence**, often denoted $\{a_n\}$, is an infinite list of its **terms** $a_1, a_2, a_3, \ldots$.

We'll define it precisely later, but one thing we are interested in is the **limit** $\lim_{n\to\infty} a_n$ (if it exists).

Here are a few first examples of sequences:

- $2, 4, 6, 8, 10, \ldots$ (that is, $a_1 = 2$, $a_2 = 4$, $\ldots$)

  This is the sequence $\{a_n\}$ with $a_n = 2n$. Clearly, $\lim_{n\to\infty} a_n = \infty$.
- $1, -1, 1, -1, 1, \ldots$

  This is the sequence $\{a_n\}$ with $a_n = (-1)^{n-1}$. The limit $\lim_{n\to\infty} a_n$ does not exist.

  [Some part of the sequence "goes to" 1 but another part to -1. There is no single value that all terms approach.]
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

  This is the sequence $\{a_n\}$ with $a_n = \frac{1}{2^n}$. Clearly, $\lim_{n\to\infty} a_n = 0$.
- $3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$

  This is the sequence $\{a_n\}$ where a_n consists of the first n (decimal) digits of π. Clearly, $\lim_{n\to\infty} a_n = \pi$.

Remark 118. In a little bit, we will also be interested in **series**. These are infinite sums such as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$. Do not confuse these two! [It would be like confusing a function and its definite integral.]

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