Trigonometric substitutions

Example 110. Everybody knows that $\cos^2 x + \sin^2 x = 1$.
Divide both sides by $\cos^2 x$ to find $1 + \tan^2 x = \sec^2 x$.

Example 111. $\int \frac{1}{\sqrt{1 - x^2}} \mathrm{d}x =$

We substitute $x = \sin\theta$ (with $\theta \in [-\pi/2, \pi/2]$ so that $\theta = \arcsin(x)$) because then $1 - x^2 = \cos^2\theta$. Since $\mathrm{d}x = \cos\theta \, \mathrm{d}\theta$, we find

$$\int \frac{\mathrm{d}x}{\sqrt{1 - x^2}} = \int \frac{\cos\theta \, \mathrm{d}\theta}{\sqrt{1 - \sin^2\theta}} = \int \frac{\cos\theta \, \mathrm{d}\theta}{\sqrt{\cos^2\theta}} = \int 1 \mathrm{d}\theta = \theta + C = \arcsin(x) + C.$$

[Note that in order to conclude $\sqrt{\cos^2\theta} = \cos\theta$, we used that $\theta \in [-\pi/2, \pi/2]$ and that $\cos\theta \geqslant 0$ for these values of θ.]

Example 112. $\int \sqrt{1 - x^2} \, \mathrm{d}x =$

We substitute $x = \sin\theta$ (with $\theta \in (-\pi/2, \pi/2)$ so that $\theta = \arcsin(x)$) because then $1 - x^2 = \cos^2\theta$. Since $\mathrm{d}x = \cos\theta \, \mathrm{d}\theta$, we find

$$\int \sqrt{1 - x^2} \, \mathrm{d}x = \int \cos^2\theta \, \mathrm{d}\theta = \ldots\text{by parts}\ldots = \frac{\theta + \sin\theta \cos\theta}{2} + C = \frac{\arcsin x + x\sqrt{1 - x^2}}{2} + C.$$

Example 113. $\int \frac{1}{1 + x^2} \mathrm{d}x =$ [We already know the answer but let's see how it comes out of trig substitution!]

We substitute $x = \tan\theta$ because then $1 + x^2 = \sec^2\theta$. Since $\mathrm{d}x = \ldots$

if you see	try substituting	because
$a^2 - x^2$ (especially $\sqrt{a^2 - x^2}$)	$x = a\sin\theta$	$a^2 - (a\sin\theta)^2 = a^2\cos^2\theta$
$a^2 + x^2$ (especially $\sqrt{a^2 + x^2}$)	$x = a\tan\theta$	$a^2 + (a\tan\theta)^2 = a^2\sec^2\theta = \frac{a^2}{\cos^2\theta}$
and, somewhat less importantly:		
$x^2 - a^2$ (especially $\sqrt{x^2 - a^2}$)	$x = a\sec\theta$	$(a\sec\theta)^2 - a^2 = a^2\tan^2\theta$

Note that (by completing the square and doing a simple linear substitution), you can put any quadratic term $ax^2 + bx + c$ into one of these three cases (for instance, $x^2 + 2x + 3 = (x + 1)^2 + 2 = u^2 + 2$ with the simple linear substitution $u = x + 1$).
This is why trigonometric substitution occurs frequently for certain kind of integrals.

Example 114. $\int \frac{1}{(1 + x^2)^2} \mathrm{d}x =$ [That's an integral we care about for partial fractions!]

Solution. We substitute $x = \tan\theta$ because ....
[After simplifying, you should find $\int \frac{\mathrm{d}x}{(1 + x^2)^2} = \int \cos^2\theta \, \mathrm{d}\theta$. Go on!]
Try to simplify your answer so your final antiderivative is $\int \frac{\mathrm{d}x}{(1 + x^2)^2} = \frac{1}{2}\left[\arctan(x) + \frac{x}{1 + x^2}\right] + C$.
[Can you see, for instance, why $\sin\theta = \frac{x}{\sqrt{1 + x^2}}$? This is simpler than $\sin\theta = \sin(\arctan(x))$, isn't it.]

Armin Straub
straub@southalabama.edu