Improper integrals

Example 93. $\int_0^\infty e^{-x}\mathrm{d}x =$

This integral is an example of an **improper integral** of type I (because one of its limits is ∞).

Make a sketch!

Replacing the upper limit with b, we have $\int_0^b e^{-x}\,\mathrm{d}x = \left[-e^{-x}\right]_0^b = 1 - e^{-b}$.

Therefore, $\int_0^\infty e^{-x}\mathrm{d}x = \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x = \lim_{b\to\infty}(1 - e^{-b}) = 1$.

Once experienced, we can just write $\int_0^\infty e^{-x}\mathrm{d}x = \left[-e^{-x}\right]_0^\infty = 0 - (-1) = 1$ and indicate that we used $\lim_{x\to\infty} - e^{-x} = 0$.

Example 94. $\int_1^\infty \frac{1}{x^4}\,\mathrm{d}x =$

Make a sketch! Your final answer should be that the integral converges and equals $\frac{1}{3}$.

Example 95. $\int_1^\infty \frac{1}{x}\,\mathrm{d}x =$

Make a sketch (looks essentially the same as in the previous example)!

$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \left[\ln|x|\right]_1^\infty$ but $\lim_{x\to\infty}\ln|x| = \infty$. Thus, the integral diverges (to ∞, in this case).

Example 96. $\int_0^1 \frac{1}{x}\,\mathrm{d}x =$

This integral is an example of an improper integral of type II (because the integrand has a vertical asymptote at one of the limits).

Make a sketch!

$\int_0^1 \frac{1}{x}\,\mathrm{d}x = \left[\ln|x|\right]_0^1$ but $\lim_{x\to 0^+}\ln|x| = -\infty$. Thus, the integral diverges (to ∞, in this case).

Example 97. $\int_{-\infty}^\infty \frac{1}{x^2+1}\,\mathrm{d}x =$

Make a sketch! (For that, note that the integrand is always positive, that it goes to 0 as $x\to\infty$ and $x\to-\infty$. Also, you can see that the maximum occurs at $x = 1$. Taken together, the graph looks like a single mound.)

Your final answer should be π.

[If necessary, review $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and its inverse function $\arctan(x)$.]