Example 58. We want to pump water out of a cubic container buried in our garden. The base of the container is 6 ft by 6 ft, and it is 4 ft deep. The top of the container is 10 ft below ground level. How much work is required to pump all the water (144 ft^3) from the container to ground level? [Water weighs 62.4 lb/ft^3.]

- Let x measure height (in ft) starting from the bottom of the container.
- We consider a (horizontal) "slice" of the container at position x (and thickness $\mathrm{d}x$).
  - The volume of this slice is $\mathrm{vol} = 36\,\mathrm{d}x$ (ft^3). Its weight is $62.4 \cdot 36\,\mathrm{d}x$ (lb).
  - This slice needs to be "lifted" $14 - x$ (ft).
  - Thus, the work for this slice is $(14 - x)\,62.4 \cdot 36\,\mathrm{d}x$ (ft-lb).
- There is slices from $x = 0$ to $x = 4$. "Adding" these up, we find that the total amount of work is

$$\mathrm{work} = \int_0^4 (14 - x)\,62.4 \cdot 36\,\mathrm{d}x = 62.4 \cdot 36 \left[14x - \frac{1}{2}x^2\right]_0^4 = 62.4 \cdot 36 \cdot 48 \approx 108{,}000 \text{ ft-lb}.$$

[Note that the answer was rounded to 3 significant digits, because we cannot expect more precision given that the weight of water per ft^3 is only given to us to 3 digits.]

Example 59.

(a) In the previous example, suppose that the container is only half full. How does the amount of work required change?

$$\mathrm{work} = \int_0^2 (14 - x)\,62.4 \cdot 36\,\mathrm{d}x \approx 58{,}400 \text{ ft-lb}$$

(b) In the previous example, suppose that the container is full but that we only pump out half of the water. How does the amount of work required change?

$$\mathrm{work} = \int_2^4 (14 - x)\,62.4 \cdot 36\,\mathrm{d}x \approx 49{,}400 \text{ ft-lb}.$$

[Note that the sum of these two is again $108{,}000$ ft-lb. Think about why that makes perfect sense!]

Example 60. (6.1.52) Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about the line $x = 1$.

[Respectively, about the line $x = 2$ for the second part of the problem.]

Example 61. Consider the solid from the previous problem sitting in your garden at ground level ($y = 0$) with x and y measured in ft. What is the amount of work required to pump water into this solid from ground level? [Water weighs 62.4 lb/ft^3.]

(a) As we did when computing the volume, we consider (horizontal) slices at height y and thickness $\mathrm{d}y$. The slice is (almost) a disk with radius $1 - y/2$ (make a sketch!!). Hence, its volume is $\pi(1 - y/2)^2\,\mathrm{d}y$ (ft^3).

[And so the total volume (asked for in the previous problem) is $\int_0^2 \pi(1 - y/2)^2\,\mathrm{d}y = \frac{2\pi}{3}$ (ft^3).]

The weight of the slice is $62.4\pi(1 - y/2)^2\,\mathrm{d}y$ (lb) and it needs to be lifted up y (ft). That takes work of $y \cdot 62.4\pi(1 - y/2)^2\,\mathrm{d}y$ (ft-lb).

"Adding" up, the total work required is $\int_0^2 y \cdot 62.4 \cdot \pi(1 - y/2)^2\,\mathrm{d}y = \ldots \approx 65.3$ (ft-lb).

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