Review 54. $\int_2^4 \frac{\mathrm{d}x}{x(\ln x)^2} = \int_{\ln(2)}^{\ln(4)} \frac{\mathrm{d}u}{u^2}$ after substituting $u = \ln(x)$.

Make sure to change the limits when substituting in a definite integral!

Example 55. We want to "lift" a 1600 kg satellite from the ground into orbit, $20,000$ km above the surface.

[These are actually typical values for a GPS satellite.]

- Initially, the satellite is sitting on the surface, about $d_1 = 6371$ km from the center of earth (for gravitation, earth behaves like all its mass is concentrated at its center).

  The goal is to bring the satellite to a distance $d_2 = 26,371$ km from the center of earth.
- The mass of the earth is $m_E = 5.973 \cdot 10^{24}$ kg. The mass of our satellite is $m_S = 1600$ kg.
- The physical law of attraction is $F = G\frac{m_S m_E}{d^2}$.

  It tells us the force of attraction between two masses (here, the satellite and earth) that are at distance d. ($G = 6.674 \cdot 10^{-11}$ Nm²/kg² is the gravitational constant.)

  [Note that this force is not constant in our problem: the values of d range from $d = d_1$ to $d = d_2$.]

  [Different sources give slightly differing values for m_E and G: these quantities are hard to measure...]

Make a sketch!!

As in Example 52, we think about the moment when the satellite is at distance x from the center of earth.

[You could also let x be the distance from the surface. That works, too, if you adjust things accordingly.]

- At that moment, the gravitational force is $G\frac{m_S m_E}{x^2}$.
- To lift up the satellite by a tiny amount of $\mathrm{d}x$, the amount of work needed is (roughly) $G\frac{m_S m_E}{x^2}\mathrm{d}x$ (force times distance).

Hence, the total amount of work is

$$\text{work} = \int_{d_1}^{d_2} G\frac{m_S m_E}{x^2}\mathrm{d}x = G m_S m_E \int_{d_1}^{d_2} \frac{\mathrm{d}x}{x^2} = G m_S m_E \left[-\frac{1}{x}\right]_{d_1}^{d_2} = G m_S m_E \left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

Plugging in our values for G, m_S, m_E, d_1, d_2, we find

$$\text{work} = (6.674 \cdot 10^{-11}) \cdot (1600) \cdot (5.973 \cdot 10^{24}) \left(\frac{1}{6371000} - \frac{1}{26371000}\right) = 7.59 \cdot 10^{10} \text{ joule}.$$

Example 56.

- What happens when we take the **limit** $d_2 \to \infty$ in the previous example? What does that mean physically?
- How does the previous problem change if the physical law of attraction was $F = G\frac{m_S m_E}{d}$? What happens now when we take the limit $d_2 \to \infty$?

Example 57. We want to pump water out of a cubic container buried in our garden. The base of the container is 6 ft by 6 ft, and it is 4 ft deep. The top of the container is 10 ft below ground level. How much work is required to pump all the water (144 ft³) from the container to ground level? [Water weighs 62.4 lb/ft³.]

[Note that the reason we again need calculus is that each drop of water needs to be "lifted" by a varying amount (ranging from 10 ft to 14 ft). Our solution will be to consider horizontal slices of the container.]