Review 27. $\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t =$

- We substitute $u = 2-\cos(t)$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = \sin(t)$ to find $\int \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int \frac{1}{u}\mathrm{d}u$

$$\int \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int \frac{1}{u}\mathrm{d}u = \ln|u| + C = \ln|2-\cos(t)| + C,$$

  and hence (by the fundamental theorem of calculus)

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \Big[\ln|2-\cos(t)|\Big]_0^\pi = \ln(3).$$

- Alternatively, once we feel comfortable with integration, we can do this substitution in a single step by also adjusting the boundaries. When $t=0$, we have $u = 2-\cos(0) = 1$, and when $t=\pi$, we have $u = 2-\cos(\pi) = 3$. Therefore,

$$\int_0^\pi \frac{\sin(t)}{2-\cos(t)}\mathrm{d}t = \int_1^3 \frac{1}{u}\mathrm{d}u = \Big[\ln|u|\Big]_1^3 = \ln(3).$$

Areas enclosed by curves

Review 28. The area between $y = \sin(x)$ and the x-axis, between $x=0$ and $x=2\pi$, is given by

$$\int_0^{2\pi} |\sin(t)|\,\mathrm{d}t = \int_0^\pi \sin(t)\,\mathrm{d}t + \int_\pi^{2\pi} -\sin(t)\,\mathrm{d}t.$$

Theorem 29. More generally, the area enclosed by the curves $y = f(x)$ and $y = g(x)$, between $x=a$ and $x=b$, is given by

$$\int_a^b [f(x) - g(x)]\,\mathrm{d}x$$

provided that $f(x) \geqslant g(x)$ (for all $x \in [a,b]$).

[Note that the area is always $\int_a^b |f(x)-g(x)|\,\mathrm{d}x$ but to work with the absolute value, we need to break the problem into subcases according to whether $f(x)-g(x) \geqslant 0$ or $f(x)-g(x) \leqslant 0$.]

Example 30. What is the area enclosed by the curves $y = \cos(x)$, $y=1$, $x=0$, $x=2\pi$?
First, write down an integral and compute its value, then look at your sketch (always make a quick sketch!) and note that your answer makes perfect geometric sense.

Example 31. We are given three curves $f(x)$, $g(x)$, and $h(x)$, intersecting in some interesting way and are asked to compute a certain enclosed area. We did that by splitting up the area into smaller regions in which we could apply Theorem 29.

Example 32. What is the area enclosed by the curves $y = 2-x^2$, $y=-x$?
(a) First, make a sketch! (Did I mention that you should always do that?)
(b) Find intersections of the curves.
(c) Write down the integral for the area of interest.
(d) Evaluate the integral.

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