Homework #5

MATH 126 — Calculus II
Friday, Sept 18

Please print your name:

Problem 1. (7.2.20) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 3x - 2y - 6.$$

Solution. Note the differential equation can be factored as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (y+3)(x-2).$$

We can now separate variables, and integrate:

$$\frac{1}{y+3}\mathrm{d}y = (x-2)\mathrm{d}x \quad \rightsquigarrow \quad \int \frac{1}{y+3}\mathrm{d}y = \int (x-2)\mathrm{d}x \quad \rightsquigarrow \quad \ln|y+3| = \frac{x^2}{2} - 2x + C$$

Hence,

$$|y+3| = e^{\frac{x^2}{2} - 2x + C} = e^C e^{\frac{x^2}{2} - 2x}.$$

We can get rid of the absolute value by replacing the positive constant e^C by a constant D which can be positive or negative. This gives us the general solution

$$y = De^{\frac{x^2}{2} - 2x} - 3.$$

[Note that the issue with the absolute value would not occur in an initial value problem, because the initial value would tell us whether (at least locally) $y > -3$ or not.] □

Problem 2. (7.2.27, Working underwater) The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{\mathrm{d}L}{\mathrm{d}x} = -kL.$$

As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

Solution. The question tells us that $L(18) = \frac{1}{2}L(0)$. The problem is to find x such that $L(x) = \frac{1}{10}L(0)$.

From the differential equation, we know that $L(x) = Ce^{-kx}$. Note that $L(0) = C$.

$L(x) = \frac{1}{10}L(0)$ becomes $Ce^{-kx} = \frac{1}{10}C$. Solving for x, we get $-kx = \ln\left(\frac{1}{10}\right)$ and hence $x = \frac{\ln(10)}{k}$. It remains to find k.

Using $L(18) = \frac{1}{2}L(0)$, we get $Ce^{-18k} = \frac{1}{2}C$, which implies $-18k = \ln\left(\frac{1}{2}\right)$ and hence $k = \frac{\ln(2)}{18}$.

Taken together, you can expect to work up to about depth

$$x = \frac{\ln(10)}{\frac{\ln(2)}{18}} = 18\frac{\ln(10)}{\ln(2)} = 18\log_2(10) \approx 59.8 \text{ ft}.$$

□

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Problem 3. (8.1.18) Evaluate the integral $\int (r^2+r+1)e^r\,\mathrm{d}r$.

Solution. We integrate by parts with $f(r)=r^2+r+1$ and $g'(r)=e^r$. With $g(r)=e^r$, we get

$$\int (r^2+r+1)e^r\,\mathrm{d}r = (r^2+r+1)e^r - \int (2r+1)e^r\,\mathrm{d}r.$$

For the new integral, we again integrate by parts with $f(r)=2r+1$ and $g'(r)=e^r$. With $g(r)=e^r$, we get

$$\int (2r+1)e^r\,\mathrm{d}r = (2r+1)e^r - \int 2e^r\,\mathrm{d}r = (2r+1)e^r - 2e^r = (2r-1)e^r.$$

Taken together,

$$\int (r^2+r+1)e^r\,\mathrm{d}r = (r^2+r+1)e^r - (2r-1)e^r = (r^2-r+2)e^r.$$

□

Problem 4. (8.4.2) Expand $\frac{5x-7}{x^2-3x+2}$ by partial fractions.

Solution. The numerator has degree less than the denominator, and the denominator factors as $(x-2)(x-1)$.

Hence, $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$ for some numbers A, B.

To find A, B, we clear denominators: $5x-7=(x-1)A+(x-2)B$.

Setting $x=2$, we find $3=A$. Setting $x=1$, we find $-2=-B$.

Hence, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$. □

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