Preparing for the Final

MATH 126 — Calculus II
Wednesday, Dec 9

Please print your name:

Problem 1. Go over all past quizzes!

Problem 2. Study the practice problems for the two midterm exams!

Problem 3. Retake the two midterm exams!

(A copy without solutions is available on our course website. Of course, you also find solutions there.)

Additional problems covering the material since the second midterm

Problem 4.

(a) Write down the Taylor series for $\sin(2x)$ at $x=0$.

(b) What is the radius of convergence of that series? What is the exact interval of convergence?

Solution.

(a) Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ (you need to know this!), we have $\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^{2n+1}$. All these series converge for all x.

(b) The radius of convergence is $R=\infty$. The interval of convergence is $(-\infty,\infty)$. □

Problem 5. Consider the series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}\, 3^n}$.

(a) What is the radius of convergence of that series?

(b) What is the exact interval of convergence?

Solution.

(a) We apply the ratio test with $a_n = \frac{(x+1)^n}{\sqrt{n}\, 3^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^{n+1}}{\sqrt{n+1}\, 3^{n+1}} \frac{\sqrt{n}\, 3^n}{(x+1)^n}\right| = \left|\frac{x+1}{3}\right| \frac{\sqrt{n}}{\sqrt{n+1}} \to \left|\frac{x+1}{3}\right| \text{ as } n\to\infty$$

Hence, the ratio test implies that $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n 3^n}$ converges if $\left|\frac{x+1}{3}\right| < 1$, that is, $|x+1| < 3$.

In particular, the radius of convergence is 3.

(b) We already know that the series converges if $|x+1|<3$, that is, if $x \in (-1-3, -1+3) = (-4, 2)$. We also know that the series diverges if $|x+1| > 3$.

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What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x=-4$ and $x=2$:

- $x=2$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(2+1)^n}{\sqrt{n}\,3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. We know that this p-series diverges.
- $x=-4$: in that case, the series is $\sum_{n=1}^{\infty} \frac{(-4+1)^n}{\sqrt{n}\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. We know that this alternating p-series converges.

The exact interval of convergence therefore is $[-4, 2)$. □

Problem 6. For each of the following series, decide whether it converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

(b) $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+3}$

(e) $\sum_{n=0}^{\infty} \frac{n!}{4^n \sqrt{n^2+3}}$

(f) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n \sqrt{n^2+3}}$

Solution.

(a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges absolutely. In fact, we know that $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

(b) $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ diverges because $(-1)^n \sqrt{n} \nrightarrow 0$ as $n \to \infty$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ converges conditionally, because $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+3}}$ diverges (by limit comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$) and $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$ converges by the alternating series test (since $\frac{1}{\sqrt{n^2+3}}$ is positive and decreases to zero).

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+3}$ converges absolutely because $\sum_{n=0}^{\infty} \frac{1}{n^2+3}$ converges (by limit comparison with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$).

(e) $\sum_{n=0}^{\infty} \frac{n!}{4^n \sqrt{n^2+3}}$ diverges because $\frac{n!}{4^n \sqrt{n^2+3}} \nrightarrow 0$ as $n \to \infty$.

(f) The series $\sum_{n=0}^{\infty} \frac{1}{4^n \sqrt{n^2+3}}$ converges by comparison with $\sum_{n=0}^{\infty} \frac{1}{4^n}$. Hence, $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n \sqrt{n^2+3}}$ converges absolutely. □

Problem 7. Let $f(x) = \frac{1}{(1-2x)^2}$.

(a) Determine $f'(x)$ as well as $\int f(x)\,\mathrm{d}x$.

(b) Use your answer for $\int f(x)\,\mathrm{d}x$ to determine the Taylor series for $f(x)$ at $x=0$.

(c) What is the radius of convergence of that series?

(d) What is the exact interval of convergence?

Solution.

(a) $f'(x) = \frac{4}{(1-2x)^3}$ and $\int f(x)\,\mathrm{d}x = \frac{1}{2} \cdot \frac{1}{1-2x} + C$

(b) By the geometric series, we have $\int f(x)\,\mathrm{d}x = \frac{1}{2} \cdot \frac{1}{1-2x} + C = \frac{1}{2}\sum_{n=0}^{\infty} (2x)^n + C = \sum_{n=0}^{\infty} 2^{n-1}x^n + C$ under the condition that $|2x| < 1$, that is, $|x| < 1/2$.

Differentiating both sides, we conclude that $f(x) = \sum_{n=0}^{\infty} 2^{n-1}nx^{n-1} = \sum_{n=1}^{\infty} 2^{n-1}nx^{n-1}$ if $|x| < 1/2$.

(c) The radius of convergence of $\sum_{n=1}^{\infty} 2^{n-1}nx^{n-1}$ is $1/2$. This is a consequence of our argument because integerating or differentiating a power series does not change the radius of convergence.

Alternatively, we can apply the ratio test to arrive at the same conclusion.

(d) We already know that the series converges if $|x| < \frac{1}{2}$, that is, if $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. We also know that the series diverges if $|x| > \frac{1}{2}$. What we don't know yet is whether the series converges at the endpoints of the interval. We still need to think about the cases $x = -\frac{1}{2}$ and $x = \frac{1}{2}$:

- $x = \frac{1}{2}$: in that case, the series is $\sum_{n=1}^{\infty} 2^{n-1}n\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} n$. This series obviously diverges.
- $x = -\frac{1}{2}$: in that case, the series is $\sum_{n=1}^{\infty} 2^{n-1}n\left(-\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1}n$. This series obviously diverges.

The exact interval of convergence therefore is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. □

Problem 8.

(a) Determine the Taylor series for $\int e^{-x^2/2}\,\mathrm{d}x$ at $x = 0$.

(b) What is the radius of convergence of that series? What is the exact interval of convergence?

Solution.

(a) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x (you need to know that!). Hence, $e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\, n!} x^{2n}$.

Integrating, we find $\int e^{-x^2/2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\, n!} \frac{x^{2n+1}}{2n+1}$.

(b) The radius of convergence is $R = \infty$ (this follows from the fact that the Taylor series for e^x converges for all x; alternatively, you can apply the ratio test). The interval of convergence is $(-\infty, \infty)$. □

Problem 9.

(a) What are the cartesian coordinates of the point with polar coordinates $r = 3$, $\theta = \frac{\pi}{6}$?

(b) What are the polar coordinates of the points with cartesian coordinates $(-2, 2)$ and $(2, -2)$?

(c) Sketch the region described by $2 \leqslant r \leqslant 4$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{4}$.

Solution.

(a) The cartesian coordinates are $x = r\cos\theta = 3\cos\frac{\pi}{6} = 3\frac{\sqrt{3}}{2}$ and $y = r\sin\theta = 3\sin\frac{\pi}{6} = \frac{3}{2}$.

(b) The polar coordinates of $(-2, 2)$ are $r = \sqrt{x^2 + y^2} = \sqrt{8} = 2\sqrt{2}$ and $\theta = \frac{3\pi}{4}$ (make a sketch!).

The polar coordinates of $(2, -2)$ are $r = \sqrt{x^2 + y^2} = \sqrt{8} = 2\sqrt{2}$ and $\theta = \frac{7\pi}{4}$ (or $\theta = -\frac{\pi}{4}$).

(c) Do it! Let me know if you have any troubles, and I will create a sketch and include it here or discuss it in our review sessions. □

Problem 10. Consider the parametric curve given by $x = t\cos(t)$, $y = t\sin(t)$ with parameter $t \in [0, 2\pi]$.

(a) Make a very rough sketch of the curve.

(b) Find the slope of the line tangent to the curve at the point corresponding to $t = \frac{\pi}{2}$.

(c) Setup an integral for the arc length of the curve.

Simplify but don't evaluate the integral. [However, if you want a real challenge, evaluate it using our techniques.]

Solution.

(a) We saw a rough sketch in class (one turn of a spiral starting at the origin and ending at $(2\pi, 0)$).

(b) The tangent line has slope $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$.

At our point, that is when $t = \frac{\pi}{2}$, the slope is $\frac{\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$.

(c) The arc length of our curve is

$$\int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^{2\pi} \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}\,\mathrm{d}t = \int_0^{2\pi} \sqrt{1 + t^2}\,\mathrm{d}t.$$

The final integral could be evaluated using $\int \sqrt{1 + t^2}\,\mathrm{d}t = \frac{1}{2}t\sqrt{1 + t^2} + \frac{1}{2}\log\left(t + \sqrt{1 + t^2}\right) + C$.

[If you want a real challenge, try to derive this indefinite integral using our techniques. But only when you are perfectly ready for the exam!] □