1 Optimization

Example 1. Find positive values of x and y that minimize $S = x + y$ if $xy = 16$.

Solution.

$S = x + y$ **(objective equation)**

$xy = 16$ **(constraint equation)**

To minimize S, we express S as a function of one variable, say, x:

Since $y = \frac{16}{x}$, we get $S(x) = x + \frac{16}{x}$.

(find min of $S(x)$) $S'(x) = 1 - \frac{16}{x^2}$

Solving $S'(x) = 0$, we find:
$$\begin{aligned} 1 - 16x^{-2} &= 0 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

Hence, $x = 4$ (see details for next problem).

Correspondingly, $y = \frac{16}{x} = 4$.

In conclusion, if $x = 4$ and $y = 4$, then S is minimized and equal to 8.

Example 2. A small rectangular garden of area 80 square meters is to be surrounded on three sides by a brick wall costing 5 dollars per meter and on one side by a fence costing 3 dollars per meter. Find the dimensions of the garden such that the overall cost is minimized.

Solution.

(setup) Let a be the length (in m) of the side with a fence,
and b the length of the other side.

Overall cost: $C = (5 + 3)a + (5 + 5)b = 8a + 10b$. **(objective equation)**

On the other hand, $ab = 80$. **(constraint equation)**

To minimize the cost C, we express C as a function of a:

Since $b = \frac{80}{a}$, we get $C(a) = 8a + 10 \cdot \frac{80}{a} = 8a + 800a^{-1}$.

(find min of $C(a)$) $C'(a) = 8 + 800 \cdot (-a^{-2}) = 8 - 800a^{-2}$

Solving $C'(a) = 0$, we find:
$$\begin{aligned} 8 - 800a^{-2} &= 0 \\ a^2 &= 100 \\ a &= \pm 10 \end{aligned}$$

Hence, $a = 10$ (see details below).

Correspondingly, $b = \frac{80}{a} = 8$.

We conclude that, to minimize costs, the length of the side with a fence should be $a = 10$ meters and the length of the other side should be $b = \frac{80}{a} = 8$ meters.

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Details. Our task was to find the absolute minimum of $C(a)$ for a in $(0, \infty)$.

We found that $a = 10$ was the only critical point in $(0, \infty)$ and hence the only candidate for a local min. To determine that there is indeed a local min at $a = 10$, we have several options:

(a) Observe that for small a (close to 0) and large a, the cost is definitely not optimal (actually the cost becomes arbitrarily large); hence, the absolute minimum must be somewhere in between, and the only candidate is $a = 10$.

(b) Apply the second-derivative test: $C''(a) = 1600a^{-3}$, so that $C''(10) = \frac{8}{5} > 0$, which shows that there is a local min at $a = 10$.

(c) Apply the first-derivative test: since, say, $C''(1) = -792 < 0$ and $C''(20) = 6 > 0$, we conclude that C' changes from $-$ to $+$ at $a = 10$, which again shows that there is a local min at $a = 10$.

Comment. We could also have expressed the cost as a function of b. Then $C(b) = 8 \cdot \frac{80}{b} + 10b = 640b^{-1} + 10b$ and $C'(b) = -640b^{-2} + 10$, so that $C'(b) = 0$ simplifies to $b^2 = 64$. We would conclude that $b = 8$ and then determine $a = \frac{80}{b} = 10$, ending up (of course!) with the same dimensions as before.

2 Optimization, business-style

Recall.

- Profit is revenue minus cost: $P = R - C$

  Here, x (in units) is a production level.

- If cost is $C(x)$ then marginal cost is $C'(x)$.
- $R = p \cdot x$ where p is price per unit.

  However, note that p will depend on x:
  $p = f(x)$ is called the **demand equation**.

Example 3. Given the cost function $C(x) = x^3 - 12x^2 + 60x + 20$, find the minimal marginal cost.

Solution.

The marginal cost function is $M(x) = C'(x) = 3x^2 - 24x + 60$.

We need to find the minimum of $M(x)$.

$M'(x) = 6x - 24$

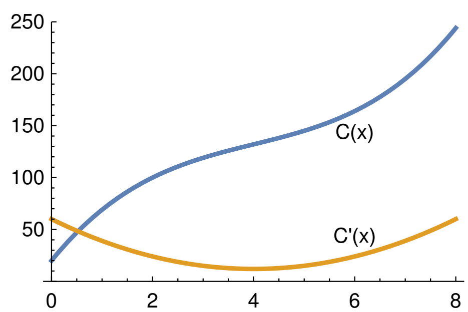
Solving $M'(x) = 0$, we find $x = 4$.

Let us check that this is a minimum:

[You could skip this step by arguing that $x = 4$ must be the minimum because it is the only candidate.]

- (second derivative test) $M''(x) = 6$
  Since $M''(4) = 6 > 0$, this is a local minimum.
- Because there is no other critical points, this must be the absolute minimum.

The minimal marginal cost is $M(4) = 12$.



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Example 4. The demand equation for a certain commodity is

$$p = x^2 - 4x + 4, \quad 0 \leqslant x \leqslant 2$$

Find the production level x and the corresponding price p that maximizes revenue.

Solution. Revenue is $R = p \cdot x = (x^2 - 4x + 4) \cdot x = x^3 - 4x^2 + 4x$.

We need to find the maximum of that $R(x)$.

$R'(x) = 3x^2 - 8x + 4$

Solving $R'(x) = 0$, we find $x = \frac{8 \pm \sqrt{64 - 48}}{6} = \frac{8 \pm 4}{6} = \frac{2}{3}, 2$.

Note that $x = 2$ is on the boundary of permitted values: indeed, for $x = 2$, the price is $p = 0$ (which clearly doesn't maximize revenue).

Hence, revenue is maximized for production level $x = \frac{2}{3}$ and price $p = x^2 - 4x + 4 = \frac{16}{9}$.

Note. If you didn't see that $x = 2$ was not a candidate, you can always perform one of the derivative tests to determine that there is a maximum at $\frac{2}{3}$ and a minimum at $x = 2$.

Example 5. At a price of 6 dollars, 50 beers were sold per night at a local cinema. When the price was raised to 7 dollars, sales dropped to 40 beers.

(a) Assuming a linear demand curve, which price maximizes revenue?

Solution. p prize per beer, x number of beer sold

Revenue is $R(x) = p \cdot x$. **(objective equation)**

Linear demand means that $p = ax + b$ (a line!) for some a, b.

We know that $(x_1, p_1) = (50, 6)$ and $(x_2, p_2) = (40, 7)$.

Hence, $p - 6 = \underbrace{\text{slope}}_{\frac{p_2 - p_1}{x_2 - x_1} = \frac{1}{-10}} (x - 50)$. This simplifies to $p = 11 - \frac{1}{10}x$. **(constraint equation)**

Revenue is $R(x) = p \cdot x = \left(11 - \frac{1}{10}x\right) \cdot x = 11x - \frac{1}{10}x^2$.

(find max of $R(x)$) $R'(x) = 11 - \frac{2}{10}x$

Solving $R'(x) = 11 - \frac{1}{5}x = 0$, we find $x = 55$.

The corresponding price is $p = 11 - \frac{1}{10} \cdot 55 = 5.5$ dollars.

(Then the revenue is $R(55) = 5.5 \cdot 55 = 302.5$ dollars.)

(b) Suppose the cinema has fixed costs of 100 dollars per night for selling beer, and variable costs of 2 dollars per beer. Find the price that maximizes profit.

Solution. p prize per beer, x number of beer sold

Cost is $C(x) = 100 + 2x$.

As before, revenue is $R(x) = 11x - \frac{1}{10}x^2$.

Profit is $P(x) = R(x) - C(x) = 9x - \frac{1}{10}x^2 - 100$.

(find max of $P(x)$) $P'(x) = 9 - \frac{2}{10}x$

Solving $P'(x) = 9 - \frac{1}{5}x = 0$, we find $x = 45$.

The corresponding price is $p = 11 - \frac{1}{10} \cdot 45 = 6.5$ dollars.

(Then the profit is $P(45) = 102.5$ dollars.)

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