$f'(a)$ is the

- **slope of the tangent line** approximating $f(x)$ at $x = a$
- **rate of change** of $f(x)$ at $x = a$

We can estimate function values by using the tangent line as an approximation:

Example 1. Suppose $f(2) = 1$ and $f'(2) = 3$.

(a) Estimate $f(2.5)$.

The tangent line at $x = 2$ is $y - 1 = 3(x - 2)$ or $y = 1 + 3(x - 2)$.

$f(2.5) \approx 1 + 3(2.5 - 2) = 2.5$

(b) Estimate $f(2.1)$.

$f(2.1) \approx 1 + 3(2.1 - 2) = 1.3$

(c) Which of the estimates do we expect to be more accurate?

The estimate for $f(2.1)$.

The tangent line at $x = 2$ is a good approximation for values of x close to 2.

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1 Slopes = rates of change

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} \quad \left(\text{i.e. } \frac{\text{change in } y}{\text{change in } x}\right)$$

Recall that we write $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ if $y = f(x)$.

- $f'(a)$ is the **rate of change** of $f(x)$ at $x = a$.
- $\frac{f(b) - f(a)}{b - a}$ is the **average rate of change** of $f(x)$ over the interval $a \leqslant x \leqslant b$. [i.e. between $x = a$ and $x = b$]

Important. If b is close to a, then $\frac{f(b) - f(a)}{b - a} \approx f'(a)$.

Example 2. Let $f(x) = x^2$.

(a) What is the average rate of change over $2 \leqslant x \leqslant 5$?

$\frac{f(5) - f(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7$

Meaning that, on average, $f(x)$ changes by 7 units per change of x by 1 unit.

(b) What is the rate of change at $x = 2$?

$f'(x) = 2x$ so that $f'(2) = 4$

(c) What is the rate of change at $x = 5$?

$f'(x) = 2x$ so that $f'(5) = 10$

Make a sketch!

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2 Higher derivatives

The derivative of the derivative is the **second derivative**.

It is denoted $f''(x)$ or $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$. Or, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Similarly, but less important, there is a third derivative and so on...

Example 3. Let $y = -2x^4 + 3x$. Find the first and second derivatives.

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = -8x^3 + 3$

(b) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -24x^2$

Example 4. Determine: $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(2x^3 - x + 1)\Big|_{x=5}$

This is the same as setting $f(x) = 2x^3 - x + 1$ and asking for $f''(5)$.

Solution.

$\frac{\mathrm{d}}{\mathrm{d}x}(2x^3 - x + 1) = 6x^2 - 1$

$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(2x^3 - x + 1) = 12x$

$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(2x^3 - x + 1)\Big|_{x=5} = 12 \cdot 5 = 60$

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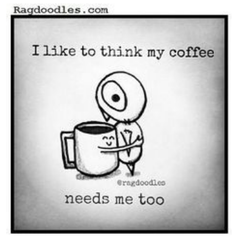
3 Back to rates of change



$f'(a)$ is the **rate of change** of $f(x)$ at $x=a$

Example 5. Suppose your fresh cup of coffee is $f(t)$ degrees (Fahrenheit) warm after t minutes.

(a) What is the meaning of $f(5)=175$?

First off, the units for $f(5)$ are degrees.

Meaning: After 5 minutes, your coffee is 175 degrees warm.

(b) What is the meaning of $f'(5)=-2$?

First off, the units for $f'(5)$ are degrees/min.

Meaning: after 5 minutes (at that moment of time), the coffee is cooling down 2 degrees/minute.

[This is the rate at which the temperature changes.]

(c) Estimate the temperature after 6 minutes.

In other words, estimate $f(6)$.

At $t=5$, the temperature is $f(5)=175$ degrees, and
it changes at a rate of $f'(5)=-2$ degrees/minute.

Hence, we estimate $f(6)\approx 175-2=173$ degrees.

Note. Mathematically, we have approximated $f(t)$ with the tangent line at $t=5$ (which has equation $f(5)+f'(5)(t-5)$).

(d) Estimate the temperature after 8 minutes.

As before, we now estimate $f(8)\approx 175-2\cdot 3=169$.

This estimate is more risky since 8 is further away from 5.

Fancy thoughts. Should we expect $f(8)<169$ or $f(8)>169$?

The rate of change should decrease as the coffee approaches room temperature. Hence, we expect that $f(8)>169$ and that $f'(8)>-2$.

Comment. We might discuss Newton's law of cooling when talking about exponential models.

(e) Given $f(5)=175$ and $f(8)=170$, what is the **average rate of change** between minute 5 and minute 8?

$$\frac{f(8)-f(5)}{8-5}=\frac{170-175}{8-5}=-\frac{5}{3}$$

Between minute 5 and minute 8, the temperature decreases on average by $\frac{5}{3}$ degrees/min.

Note. This is an average rate of change!

$f'(5)=-2<-\frac{5}{3}$ and we expect that $f'(8)$ is in $\left(-\frac{5}{3},0\right)$.

4 Marginal cost/revenue/profit

- If $C(x)$ is the cost to produce x units, then
- $C'(x)$ is the **marginal cost** (at production level x).

Marginal cost is measured in cost/unit.

It is the cost per (additional) unit at production level x.

Note that $C'(x) \approx \frac{C(x+1) - C(x)}{1}$.

The right-hand side is literally the cost to produce one more item. However, it is beneficial to also allow fractional units, in which case $C'(x)$ is more appropriate.

Example 6. Suppose the cost (in dollars) of producing x units of a product is given by $C(x) = \text{secret}(x)$ dollars.

(a) What is the cost of producing 50 units?

$C(50)$ dollars

(b) What is the marginal cost when the production level is 50 units?

$C'(50)$ dollars/unit

(c) At what level of production is the marginal cost 100 dollars/unit?

Need to solve $C'(x) = 100$.

Each such x is a level of production when the marginal cost is 100 dollars/unit.

(There could be several such levels x of production.)

(d) How many units can we produce with 1000 dollars?

Need to solve $C(x) = 1000$.

Then x is the number of units can we produce with 1000 dollars.

Profit is revenue minus cost: $P(x) = R(x) - C(x)$.

As before, x is the production level.

Marginal revenue and marginal profit are likewise defined:

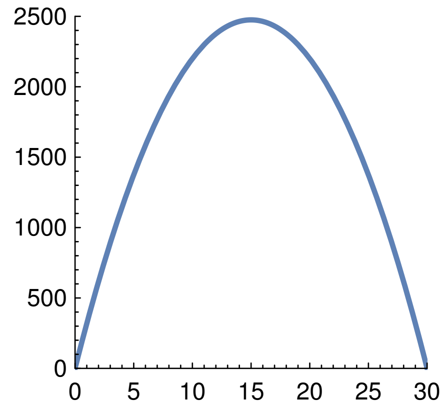
- **Marginal revenue** is $R'(x)$.

  This is the (extra) revenue for an additional unit (at production level x).

- **Marginal profit** is $P'(x)$.

  This is the (extra) profit for an additional unit (at production level x).

Example 7. Suppose $s(t)$ is the height in miles after t minutes of a rocket that is shot up vertically.



(a) What is the meaning of $s(5) = 1375$?

First off, units: $s(5)$ is miles.

After 5 minutes, the rocket is 1375 miles high.

(b) What is the meaning of $s'(5) = 220$?

First off, units: $s'(5)$ is miles/min.

After 5 minutes, the rocket has a speed of 220 miles/min (13200 miles/h).

(c) What is the meaning of $s''(5) = -22$?

First off, units: $s''(5)$ is (miles/min)/min, or miles/min^2.

After 5 minutes, the rocket has an acceleration of -22 miles/min^2.

Physics comment. Earth's gravitation is about 22 miles/min^2 (or 32.2 ft/sec^2). In other words, our rocket is ballistic (only initially powered, then in free fall).

(d) When is the altitude of the rocket 2000 miles?

To find such a time t, we need to solve $s(t) = 2000$.

[The picture suggests $t \approx 8.5$ and $t \approx 21.5$.]

(e) When does the rocket land again?

To find that time, we need to solve $s(t) = 0$.

One solution is $t = 0$ but we are looking for the other one.

[The picture suggests $t = 30$.]

(f) What is the maximal height the rocket reaches?

To find the time t of maximal height, we need to solve $s'(t) = 0$.

[The picture suggests $t = 15$ and a maximal height of $s(15) \approx 2500$ miles.]

Just for fun. These numbers are all made up. However, they are (in some aspects) not too far off from the 2017/7/28 launch of a North Korea missile. That missile reached a height of about 2315 miles and landed after 47 minutes.

https://en.wikipedia.org/wiki/Hwasong-14

For comparison, the ISS is 205-270 miles above earth, the moon $238,900$ miles.

Example 8. Solve the last three parts of the previous problem if $s(t) = 330t - 11t^2$.

(a) When is the altitude of the rocket 2000 miles?

To find such a time t, we need to solve $s(t) = 2000$.

$330t - 11t^2 = 2000$, that is, $-11t^2 + 330t - 2000 = 0$
has the two solutions $t = \frac{-330 \pm \sqrt{330^2 - 4(-11)(-2000)}}{-22} = 8.429, 21.571$.

(b) When does the rocket land again?

To find that time, we need to solve $s(t) = 0$.

$\underbrace{330t - 11t^2}_{= t(330 - 11t)} = 0$ has the solutions $t = 0$ and $t = \frac{330}{11} = 30$.

As suggested by the graph, the rocket lands at $t = 30$.

(c) What is the maximal height the rocket reaches?

To find the time t of maximal height, we need to solve $s'(t) = 0$.

$s'(t) = 330 - 22t = 0$ has the solution $t = \frac{330}{22} = 15$.

Thus, the maximal height is $s(15) = 2475$ miles.

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