(chain rule)

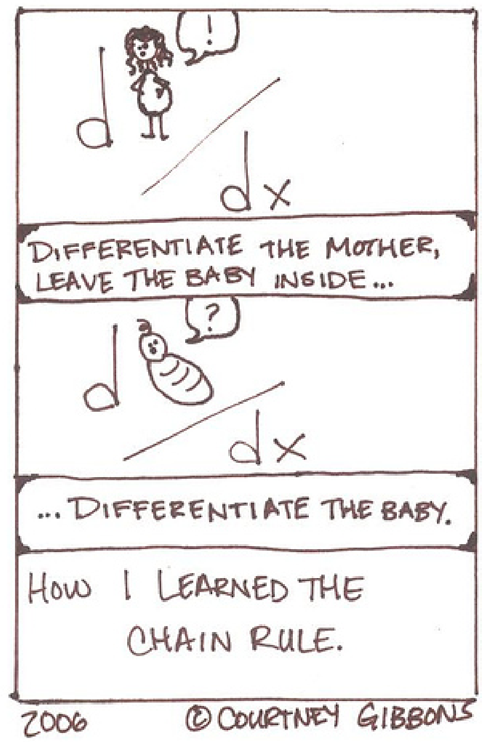
$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

(product rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)\,g'(x)$$

(quotient rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$



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1 Chain rule

Example 1. (warmup) If $f(x) = x + \sqrt{x+1}$ and $g(x) = x^4 + 1$,

then $f(g(x)) = g(x) + \sqrt{g(x)+1} = x^4 + 1 + \sqrt{x^4+2}$.

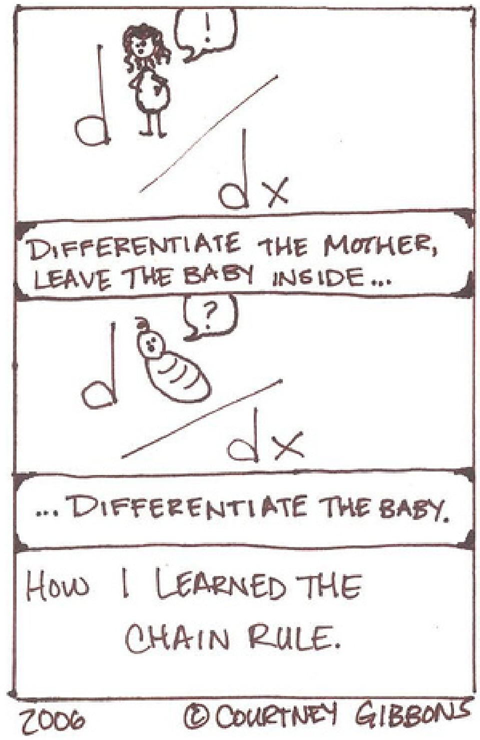
(chain rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Why?

In short form: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}x}$

(Here, $z = g(x)$ and $y = f(z) = f(g(x))$.)



Example 2.

(a) Write $h(x) = (x^3 + 7)^{10}$ in the form $f(g(x))$.

(b) Differentiate $h(x) = (x^3 + 7)^{10}$.

Solution.

(a) The natural choice is: $f(x) = x^{10}$ and $g(x) = x^3 + 7$

(b) First, we compute $f'(x) = 10x^9$ and $g'(x) = 3x^2$.

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= 10 g(x)^9 \cdot (3x^2) \\ &= 10(x^3+7)^9 \cdot (3x^2) \\ &= 30x^2(x^3+7)^9 \end{aligned}$$

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Example 3.

(a) Write $h(x) = \sqrt{x^2 - 3\sqrt{x}}$ in the form $f(g(x))$.

(b) Differentiate $h(x) = \sqrt{x^2 - 3\sqrt{x}}$.

Solution.

(a) The natural choice is: $f(x) = \sqrt{x}$ and $g(x) = x^2 - 3\sqrt{x}$

(b) First, we compute $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = 2x - \frac{3}{2}x^{-1/2}$.

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= \tfrac{1}{2}g(x)^{-1/2} \cdot \left(2x - \tfrac{3}{2}x^{-1/2}\right) \\ &= \tfrac{1}{2}(x^2 - 3\sqrt{x})^{-1/2} \cdot \left(2x - \tfrac{3}{2}x^{-1/2}\right) \end{aligned}$$

The chain rule applied with $f(x) = x^r$ results in:

(generalized power rule) $\frac{\mathrm{d}}{\mathrm{d}x}[g(x)^r] = r\,g(x)^{r-1} \cdot g'(x)$

2 Product and quotient rule

(product rule) $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)\, g'(x)$

Example 4. Differentiate $h(x) = (x^2+3)(2x^4-1)$.

Solution. (by multiplying out)

$h(x) = 2x^6 + 6x^4 - x^2 - 3$

$h'(x) = 12x^5 + 24x^3 - 2x$

Solution. (via product rule)

Write $h(x) = f(x)g(x)$ with $f(x) = x^2+3$ and $g(x) = 2x^4-1$.

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)\, g'(x) \\ &= (2x)(2x^4-1) + (x^2+3)(8x^3) \\ &= (4x^5-2x) + (8x^5+24x^3) \\ &= 12x^5 + 24x^3 - 2x \end{aligned}$$

Example 5. Differentiate $h(x) = x^2(x^3+7)^{10}$.

(Multiplying out is still possible, but would be a huge pain.)

Solution.

Write $h(x) = f(x)g(x)$ with $f(x) = x^2$ and $g(x) = (x^3+7)^{10}$.

Clearly, $f'(x) = 2x$. We computed $g'(x)$ earlier:

$g'(x) = 10(x^3+7)^9 \cdot 3x^2 = 30x^2(x^3+7)^9$ (chain rule)

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)\, g'(x) \\ &= 2x(x^3+7)^{10} + x^2 \cdot 30x^2(x^3+7)^9 \\ &= 2x(x^3+7)^{10} + 30x^4(x^3+7)^9 \qquad \text{(fine final answer)} \\ &= (2x(x^3+7) + 30x^4)(x^3+7)^9 \\ &= (32x^4+14x)(x^3+7)^9 \end{aligned}$$

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(quotient rule) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Example 6. $h(x) = \frac{1}{x}$ two ways

Solution. (power rule)

Since $h(x) = x^{-1}$, $h'(x) = -x^{-2} = -\frac{1}{x^2}$.

Solution. (quotient rule)

Write $h(x) = \frac{f(x)}{g(x)}$ with $f(x) = 1$ and $g(x) = x$.

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}$$

Example 7. Differentiate $h(x) = \frac{x^2 - 2}{3x + 7}$.

Solution.

Write $h(x) = \frac{f(x)}{g(x)}$ with $f(x) = x^2 - 2$ and $g(x) = 3x + 7$.

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(2x) \cdot (3x + 7) - (x^2 - 2) \cdot 3}{(3x + 7)^2} \\ &= \frac{(6x^2 + 14x) - (3x^2 - 6)}{(3x + 7)^2} \\ &= \frac{3x^2 + 14x + 6}{(3x + 7)^2} \end{aligned}$$