Nearly done!

- You still have until the end of Dec 1 to complete:
  - all Chapter 5&6 assignments and quizzes,
  - the Chapter 5&6 online test.

    (See Slides #11 for what to expect.)

- You can compute your course grade yourself:

  $\text{grade} = 0.3 \cdot \text{homework} + 0.3 \cdot \text{quizzes} + 0.32 \cdot \text{tests} + 0.08 \cdot \text{final}$

  If you have at most 2 unexcused absences, the final exam score will replace the lowest of your four test scores.

- Today: review and more info on final exam

You have until Dec 6 to take the online final exam. (See next slide for what to expect.)

  There will be no in-class final exam.

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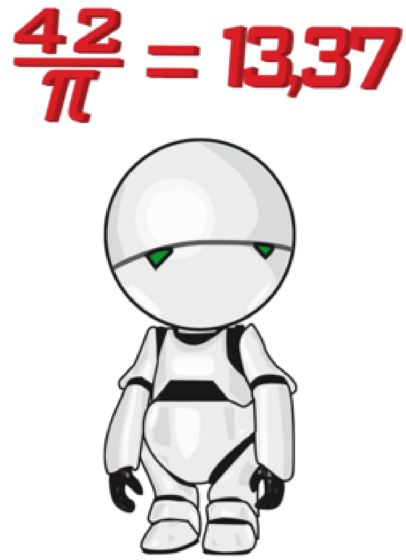
1 What to expect on the final exam

You have until Dec 6 to take the online final exam.

The password is: 42

You have 120min for 16 questions.

As on the 3rd online test, please be careful when following the **rounding** instructions.



by Carcass @ MySoti

- Earlier material from Chapters 1–4:
  - determine equation of a tangent line

    [If MLP asks for an equation, it needs to have an "$=$". For instance, something like $y = 5(x-2)+7$]
  - differentiate polynomial
  - given graph, tabulate sign of $f(x)$, $f'(x)$, $f''(x)$
  - given cost function, minimize marginal cost
  - optimize ticket price
  - differentiate something like $x^8 e^{-3x} - \sqrt{x} \cdot \ln(x)$
  - differentiate something like $\sqrt{e^{2x}+19}$
  - find the min/max of a function
- Current material from Chapters 5&6:
  - exponential growth of insect population
  - continuous interest
  - elasticity of demand
  - compute an antiderivative
  - compute an integral like $\int_1^5 \left(2x^3 - \frac{1}{x}\right) \mathrm{d}x$
  - area under, say, $y = \frac{3}{x} + \sqrt{x} + 1$ from $x = 1$ to $x = 4$
  - compute average value of a function
  - compute consumer's surplus

Example 1. Find the local extrema of $f(x) = (6 + x)e^{-4x}$. Classify them as min/max.

Solution.

- To find the critical points, we solve $f'(x) = 0$.

  $$\begin{aligned} f'(x) &= 1 \cdot e^{-4x} + (6 + x) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} e^{-4x}\right) && \text{(product rule)} \\ &= e^{-4x} + (6 + x) \cdot (-4e^{-4x}) && \text{(chain rule)} \\ &= (-23 - 4x)e^{-4x} \end{aligned}$$

  $(-23 - 4x)e^{-4x} = 0$

  $-23 - 4x = 0$

  Hence, $x = -\frac{23}{4}$ is the only critical point.

- We need to decide if there is a min/max at $x = -\frac{23}{4}$

  $\Longrightarrow$ first-derivative test or second-derivative test

  The first-derivative test is a bit easier here (but trickier?):

  The sign of $f'(x)$ is the same as the sign of $-23 - 4x$.

  The latter is a line, sloped downwards.

  $\Longrightarrow$ $f'(x)$ changes from $+$ to $-$ at $x = -\frac{23}{4}$.

  $\Longrightarrow$ $f(x)$ has a max at $x = -\frac{23}{4}$.

Example 2. Carry out the second-derivative test.

Solution. $f'(x) = (-23 - 4x)e^{-4x}$

$$\begin{aligned} f''(x) &= -4 \cdot e^{-4x} + (-23 - 4x) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x} e^{-4x}\right) && \text{(product rule)} \\ &= -4e^{-4x} + (-23 - 4x) \cdot (-4e^{-4x}) && \text{(chain rule)} \\ &= (88 + 16x)e^{-4x} \end{aligned}$$

In particular, $f''(-\frac{23}{4}) = (88 - 4 \cdot 23)e^{23} = -4e^{23} < 0$.

Hence, $f(x)$ has a max at (the critical point) $x = -\frac{23}{4}$.

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3 Review: Optimizing

Example 3. At a price of 6 dollars, 50 beers were sold per night at a local cinema. When the price was raised to 7 dollars, sales dropped to 40 beers.

(a) Assuming a linear demand curve, which price maximizes revenue?

Solution.

(setup) p prize per beer, x number of beer sold

Revenue is $R(x) = p \cdot x$. **(objective equation)**

Linear demand means that $p = ax + b$ (a line!) for some a, b.

We know that $(x_1, p_1) = (50, 6)$ and $(x_2, p_2) = (40, 7)$.

Hence, $p - 6 = \underbrace{\text{slope}}_{\frac{p_2 - p_1}{x_2 - x_1} = \frac{1}{-10}} (x - 50)$.

This simplifies to $p = 11 - \frac{1}{10}x$. **(constraint equation)**

Revenue is $R(x) = p \cdot x = \left(11 - \frac{1}{10}x\right) \cdot x = 11x - \frac{1}{10}x^2$.

(find max of $R(x)$) $R'(x) = 11 - \frac{2}{10}x$

Solving $R'(x) = 11 - \frac{1}{5}x = 0$, we find $x = 55$.

The corresponding price is $p = 11 - \frac{1}{10} \cdot 55 = 5.5$ dollars.

(Then the revenue is $R(55) = 5.5 \cdot 55 = 302.5$ dollars.)

(b) Suppose the cinema has fixed costs of 100 dollars per night for selling beer, and variable costs of 2 dollars per beer. Find the price that maximizes profit.

Solution.

(setup) p prize per beer, x number of beer sold

Cost is $C(x) = 100 + 2x$.

As before, revenue is $R(x) = 11x - \frac{1}{10}x^2$.

Profit is $P(x) = R(x) - C(x) = 9x - \frac{1}{10}x^2 - 100$.

(find max of $P(x)$) $P'(x) = 9 - \frac{2}{10}x$

Solving $P'(x) = 9 - \frac{1}{5}x = 0$, we find $x = 45$.

The corresponding price is $p = 11 - \frac{1}{10} \cdot 45 = 6.5$ dollars.

(Then the profit is $P(45) = 102.5$ dollars.)

4 Review: Integrals

$\int_a^b f(x) \mathrm{d}x = F(b) - F(a)$, where F is an antiderivative of f.

This is "the integral of $f(x)$ from $x = a$ to $x = b$".

It is common to write $\left[F(x)\right]_a^b = F(b) - F(a)$.

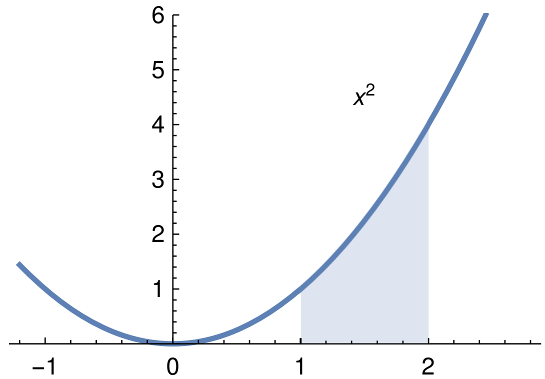
4.1 Applications

- $\int_a^b f(x) \mathrm{d}x$ is the **area** under the graph of $f(x)$ between a and b

  For instance. The previous example shows that the shaded area is

  $$\int_1^2 x^2 \mathrm{d}x = \frac{7}{3}$$

  units.



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- $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ is the **average value** of $f(x)$ between a and b
- The **consumer's surplus** at sales level $x = A$ for a commodity with demand curve $p = f(x)$ is

$$\int_0^A [f(x) - f(A)]\mathrm{d}x.$$

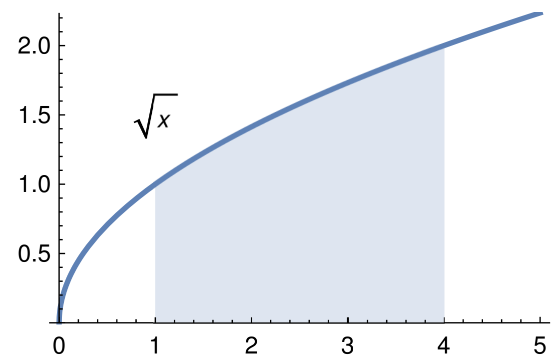
Example 4. Evaluate $\int_1^2 x^2 \mathrm{d}x$.

Solution. $\int_1^2 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^2 = \frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}$

Example 5. Determine the area under the curve $y = \sqrt{x}$ from $x = 1$ to $x = 4$.

Solution.

$$\begin{aligned}\int_1^4 \sqrt{x}\mathrm{d}x &= \left[\frac{2}{3}x^{3/2}\right]_1^4 \\ &= \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2} \\ &= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}\end{aligned}$$



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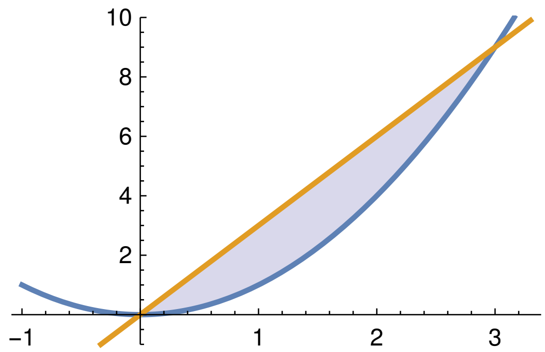
Example 6. Find the area of the region enclosed by the curves

$$y = x^2, \quad y = 3x.$$

Solution. First, make a sketch!

Doing so, we see that the area is:

$$\int_0^3 3x\,\mathrm{d}x - \int_0^3 x^2\,\mathrm{d}x = \ldots = \frac{27}{2} - 9 = \frac{9}{2}$$



Example 7. Find the average value of the function $f(x) = x^2 + 1$ from $x = 0$ to $x = 3$.

Solution. $\frac{1}{3-0}\int_0^3 (x^2+1)\mathrm{d}x = \ldots = 4$

[Make a sketch of $f(x)$ and that the function increases from $f(0) = 1$ to $f(3) = 10$. If the graph was a line, the average value would be $\frac{1+10}{2} = 5.5$. Here, the average is less because the function "spends more time" at small values.]

Example 8. During a certain 24 hour period, the temperature at time t (measured in hours from the start of the period) was

$$T(t) = 43 + 9t - \frac{1}{2}t^2$$

degrees. What was the average temperature during that period?

Solution. The average temperature was

$\frac{1}{24-0}\int_0^{24}\left(43 + 9t - \frac{1}{2}t^2\right)\mathrm{d}t = \ldots = 55$ degrees.

Example 9. Find the consumer's surplus for the demand curve $p = 5 - \frac{x}{6}$ at the sales level $x = 12$.

Solution. Recall that the demand curve describes the relationship between price p and the quantity x of products that can be sold at that price ("demand").

The consumer's surplus at sales level A for a commodity with demand curve $p = f(x)$ is

$$\int_0^A [f(x) - f(A)]\mathrm{d}x.$$

Hence, here, consumer's surplus is

$$\int_0^{12} \left[\left(5 - \frac{x}{6}\right) - 3\right]\mathrm{d}x = \ldots = 12.$$

What does consumer's surplus measure? If we want to sell A products at once ("in an open market"), the price needs to be set as $f(A)$ for a total revenue of $A \cdot f(A) = \int_0^A f(A)\mathrm{d}x$.

If a company wanted to get the most out of its customers, it could (in a non-open market) start by asking a very high price and selling just a few products, then lower the price a bit and so on.

Mathematically, the company would sell products in batches of Δx. Let $x_1, x_2, x_3, \ldots$ be the total number of products after selling $1, 2, 3, \ldots$ such batches (so $x_j = j \cdot \Delta x$). The amount of money paid by the consumers would be

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots$$

(This is probably easiest to see when looking at the graph of a specific demand curve.) It then becomes clear that, when taking smaller and smaller Δx, this amount of money approaches

$$\int_0^A f(x)\mathrm{d}x.$$

The consumer's surplus is the difference of this "extorted" amount and the amount in an open market:

$$\int_0^A f(x)\mathrm{d}x - A \cdot f(A) = \int_0^A [f(x) - f(A)]\mathrm{d}x$$